THE FUNDAMENTAL IDEA OF MATHEMATICAL TASK DESIGN IN CHINA: ORIGIN AND DEVELOPMENT

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Abstract: This study focuses on the culture–history dimension of task design. Different cultures with their own advantages and disadvantages are ，rather than oppositional，complementary. This study attempts to systemically articulate the fundamental idea of an "indigenous" task design in China—"problem variation" and its underlying curriculum framework (two bases and design with variation), its philosophical origin in I Ching, its historical roots in Jiu Zhang Suan Shu, its exemplified analysis ,its "indigenous" model , and its recent developments. This research enables us to comprehend the fundamental idea of mathematical task design from, rather than well-known deductive tradition, the relation-oriented inductive tradition in China.

Keywords: Problem variation; I Ching; JiuZhang SuanShu; Two bases

Different curriculum traditions are developed in different cultural communities (for a general discussion, see Xie & Carspecken, 2008). Contextualization problems, which focus on facilitating connections between mathematics and situations, are regarded as the common curricular trend in the Western curriculum (Clarke, 2006). By contrast, problem variations, which focus on facilitating connections between concepts and methods, play an important role in the Eastern curriculum. Problem variation is perceived as one of the most valuable experiences within mathematics education community in China (e.g., Sun, 2011a；2011b). From culture perspective, Tse, Marton, Ki, and Loh (2010) argue that variation practice stems from the Chinese language expression. Sun (2011b) examined cultural grounds of variation practice, including examination, the national curricula goal, and pedagogy to cater to individual differences. In this paper, we attempt to systemically articulate the fundamental idea of problem variation as an indigenous task design in China, its underlying curriculum framework (two bases and teaching design with variation), its philosophical origin in I Ching, its historical roots in Jiu Zhang Suan Shu, and its recent developments.

Generally, there are two main mathematical traditions in the world. One is the Greeks' deductive tradition, which focuses on geometry. Another is the inductive tradition, which focuses on arithmetic and algebra, from Egypt, Babylon, Israel, China, and India. These two traditions reflect two different types of mathematical education traditions. This study will address the latter, mathematical education tradition in China rarely known.

TWO TREASURES IN CHINESE TASK DESIGN: TWO BASES AS GOAL AND TEACHING DESIGN WITH VARIATION AS INSTRUMENT

Similar to the traditional Chinese martial arts and calligraphy education, Chinese mathematics education has been emphasizing the two bases as local own curriculum guidance, which guides local curriculum design and instruction as its national curriculum outline/standards in the whole country from the Ministry of Education in (1963; 1999, 2001). Coincidently, the Chinese local instruction is always described as "teaching with variation" (变式教学) by Chinese experts (e.g., Gu, Huang, Marton, 2004; Huang, 2002). Here, "two bases" denotes the importance of *invariant elements*, whereas "teaching with variation" denotes the value of *variant elements*. Obviously, the invariant (double bases) and variation elements (变式 bianshi teaching) are regarded as the hidden task design framework in China. Why is this framework emphasized in China? Why is the framework de-emphasized in other countries?

CHINESE PHILOSOPHICAL ORIGIN: I CHING

In ancient Chinese philosophy, the concept of yin-yang (simplified Chinese: 阴阳; traditional Chinese: 陰陽; pinyin: yīnyáng), is used to describe how seemingly opposite or contrary forces are interconnected and interdependent in the natural world; and, how they give rise to each other as they interrelate to one another. This concept lies at the origins of many branches of classical Chinese science and mathematics , as well as being a primary guideline of traditional Chinese medicine (Porkert, 1974). It can be traced back to the philosophical origin of the ancient Chinese culture, I Ching (易经), also known as the Classic of Changes, the first of the Five Classics by Confucius. The systematic ideas with variation: the dynamic balance of opposites, the evolution of events as a process, and acceptance of the inevitability of change, was systematically presented and was centered in I Ching (Hua, 1999). The variant -invariant elements idea above has heavily shaped Chinese science and culture and acted as a holistic fundamental thought in mathematics, science, medicine, cosmology, and philosophy from the time of the Zhou Dynasty (1122 BC – AD 256)(Needham, 1959) . The local world view, "nature remains essentially the same despite all apparent changes" (萬變不離其宗) and local belief that a good design should play a role to abstract the invariant concepts from a varied situation and apply these invariant concepts to varied situations (變中發現不變, 以不變應萬變) embodied this influence.

CATEGORIZATING MODEL: AN ANCIENT CHINESE THOUGHT OF TASK DESIGN USING JIU ZHANG SUAN SHU AS THE IDEOLOGICAL SOURCE

The ideas stressing invariant–variant concept above appeared in the preface below as a central spirit of guiding liuhui`s commentary on Jiu Zhang Suan Shu (JZSS) before A.D. the 3rd century, a more than 2000-year-old Chinese textbook, which has played similar role to the Euclid`s elements in Asian countries.

…Although they (knowledge tree) are diverse, those branches grow from same root (故枝條雖分而同本榦者，知發其一端而已)

………….in the preface in liuhui`s commentary (Guo, 1984, Page 7).

The invariant - variation relationship is represented as the thought of categorization in JZSS. It is described as the ideology of "grasping ways beyond categories, categorizing in order to unite categories (以法通類, 以類相從)" in ancient China. The thought of categorizing was illustrated by classifying 246 variant problems into 9 categories (歸類) below:

- 1. Fangtian (方田) - rectangular fields;
- 2. Sumi (粟米) - millet and rice. Exchange of commodities at different rates; pricing;
- 3. Cuifen (衰分) - proportional distribution. Distribution of commodities and money at proportional rates;
- 4. Shaoguang (少广) - the lesser breadth. Division by mixed numbers;
- 5. Shanggong (商功) - consultations on works. Volumes of solids of various shapes;
- 6. Junshu (均输) - equitable taxation;
- 7. Yingbuzu (盈不足) - excess and deficit. Linear problems solved by using the principle known later in the West as the "rule of false position";
- 8. Fangcheng (方程) - the rectangular array. Systems of linear equations;
- 9. Gougu (勾股) - base and altitude. Problems involving the principle known in the West as the "Pythagorean theorem."

Since the emergence of JZSS, the categorizing thought has characterized as the categorization model for mathematical task design by their context from mathematical application tradition, which play a role as an associated pedagogy of JZSS.

CATEGORIZATION MODEL HAS BECOME AN IMPORTANT PRINCIPLE FOR TASK DESIGN AND HAS PLAYED AN IMPORTANT ROLE IN THE HISTORY OF CHINESE MATHEMATICS EDUCATION

In Chinese history, the categorizing model aimed to achieve the goal of Shu (术) (similar to the general methods) in problem-oriented tradition from Oriental mathematics: "… to produce new methods from a category of problems, promote them up to the level of general method, generalize them into Shu (术), and deploy these Shu to solve various similar problems, which are more complicated, more important, and more abstruse" (Wu & Li, 1998). Almost all problems were categorized into several categories in Wucao classics mathematics book (五曹算經) and Xiahouyang classics mathematics book (夏侯陽算经) in ancient China (Wang, 1996). Before the Chinese curriculum was imported from the Western system, the categorizing model has been an unspoken task design framework. For example, mathematical problems grouped into the following categories were a typical practice of Chinese curriculum (Wang, 1996).

1. Difference/sum category;
2. Speed category,
3. Tree-planting category;
4. Age category;
5. Availing category;
6. Engineering category;
7. Profit category.

From Greek`s logic tradition perspective, axiomatic approach and rigorous proofs in Euclid`s element remain the cornerstone of mathematics in the West. Accordingly, definition/theorem-based model stressing content knowledge theme gradually formed the fundamental idea of mathematics task organization/ design in the West and has played an important role in the history of western mathematics education, where word problems, labelled as "application problems" ("应用题"), played a role of knowledge application. By contrast, problem-solving approach and application mathematics in JZSS remains the cornerstone of mathematics in the East. Its associated categorization model in JZSS gradually formed the fundamental idea of mathematics task organization/ design in China. It is interesting to note this model stresses an category-based inductive tradition, rather than the definition/theorem-based deductive tradition in the West, where word problems played a role in relations-oriented knowledge introduction (Bartolini , Sun, Alessandro, 2013).

PROBLEM VARIATION: FROM A SINGLE PROBLEM TO A CATEGORY OF PROBLEMS

The tradition of categorizing was not implemented until the Chinese mathematics curriculum was imported from the West in 1878 (Wang, 1996). However, an associated pedagogy stressing the process of categorization was developed in China after 1878. This pedagogy centered on the idea of expanding a single problem to a category of problems with variation. It also aims to establish the necessary and sufficient conditions to determine each category of problem set, which is similar two important parameters of mathematics structure, the dimensions of possible variation and the associated ranges of permissible change pointed out by the study (Watson and Mason , 2005, 2006). This practice is called bianshi (變式) in Chinese, where *bian* stands for "changing" and *shi* means "form." Although it has spread into a wide range and variety of forms in China (Sun, 2007), "indigenous" variation practice is mainly applied to mathematics subjects only, which is different from the variation theory applied to all other subjects. This practice refers to the "routine" daily practice commonly accepted by Chinese teachers, the local experience used broadly in example or exercise design to extend the original examples, known widely in a certain way as "one problem multiple solutions" (OPMS, 一題多解, varying solutions), "one problem multiple changes" (OPMC, 一題多變, varying conditions and conclusions), and "multiple problems one solution" (MPOS, 多題一解, varying presentations) (Sun, 2007; 2011a). This practice, appearing rarely in the West, is typically regarded as a natural strategy to deepen the understanding in local curriculum as a daily routine, which perhaps makes the "indigenous" practice distinctive. This strategy, easily traced to any single teaching material (such as textbooks or teaching plans) at school and any single learning material (such as student exercises or worksheets) done after school in China. As mentioned before, it seemed that Chinese arithmetic development, textbooks, their textbook reference books, and particular variation practices, might be a good clue for understanding Chinese mathematics education system rarely known outside of Chinese community.

A TYPICAL EXAMPLE

In the following, I will illustrate them with a typical textbook example in the theme C (Sun, Teresa, Loudes, 2013) in ICMI study 22. I will focus on one of them in a Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005) that has been used for over 30 years by the majority of Chinese students from diverse backgrounds. It represents the Chinese national curriculum and is seen as an authoritative guide on what to teach/learn. To grasp its distinctiveness, a comparison between Chinese and American textbooks is carried out. The Portuguese textbook (Gregório, Valente, & Calafate, 2010) with supporting teacher guide (similar to the Chinese teacher guide in that it informs teachers about appropriate goals and pedagogies) was chosen as a mirror. Here we choose textbook because China has a clear textbook-centred tradition. Textbooks play multiple functions in the Chinese mathematical education system, such as: as tools for teachers` professional development by studying textbooks (e.g. Ma, 1999); as self-learning instruments for out-of-school learners; and as the main medium for teaching and learning in the classroom. In these ways, textbooks play a central role in shaping students' learning and teachers' teaching. Addition and subtraction was chosen textbook because it is vital and central concepts for later mathematic learning, which would influence numeracy, algorithm understanding of multi-digit addition and subtraction, of multiplication and division, of decimals, of fractions. It is central to developing number sense and is also the basis for the four fundamental operations on numbers and concepts that comprise elementary school mathematics (Sun, 2013a). Not only does it connect to all important concepts, it is also a prerequisite for any real understanding of whole and rational number system.

Different features of task design for addition and subtraction from Chinese / Portuguese textbooks

The content structure of Chinese textbooks is fixed. The organization is consistent without repetition:

knowing numbers 1-5 as a foundation, then knowing addition and subtraction algorithms from 1-5;

knowing numbers 6-10, then addition and subtraction algorithms from 6-10;

knowing numbers 11-20, then addition and subtraction algorithms from 11-20;

knowing numbers within100 (1000,10000), then addition and subtraction algorithms within100(1000,10000), step by step.

It is interesting to note that addition or subtraction is not introduced directly, but its knowledge foundation as knowing number by OPMC is systematically provided. Fig. 1, 2 shows two examples introducing the quantity concept of 4, 6, 7, called cardinal number, by the problem variations with composition and decomposition concept connection in the Chinese textbook.

It is noteworthy that the design is unique that knowing number, concept of addition, and concept of subtraction are united together in all 6 chapters and gradually expand from 0-5, 6-10, 11-20, two-digit, three-digit, above four-digit in the Chinese textbook, which is separated into 15 chapters with tittles of pattern and number sense, that of addition strategy, that of subtraction strategy in the Portuguese textbook. Their design goals and pedagogies of figure 1& 2 are explained in the following in its reference book.

> Knowing numbers is the premise of calculation. Conversely, calculation will help to deepen understanding of numbers. For young children, the strategy combining

> knowing number with basic calculations would be, not only easy for learning number concept, but also conductive to consolidate basic calculations learned inversely. (Elementary Mathematic Department, 2005, P.34)

The goal and pedagogy of figure 2 is explained in the following in Chinese reference book.

> The teaching should follows the following procedure: counting → understanding of the order of number → comparison of two adjacent numbers → writing digit →order of number →composition and decomposition of number. The composition and decomposition of number is the focal point. This arrangement, on one hand, reflects the rich meaning of number concept, on the other hand, also reflects logical order of knowing number as foundation of basic calculations (Elementary Mathematic Department, 2005, 67).

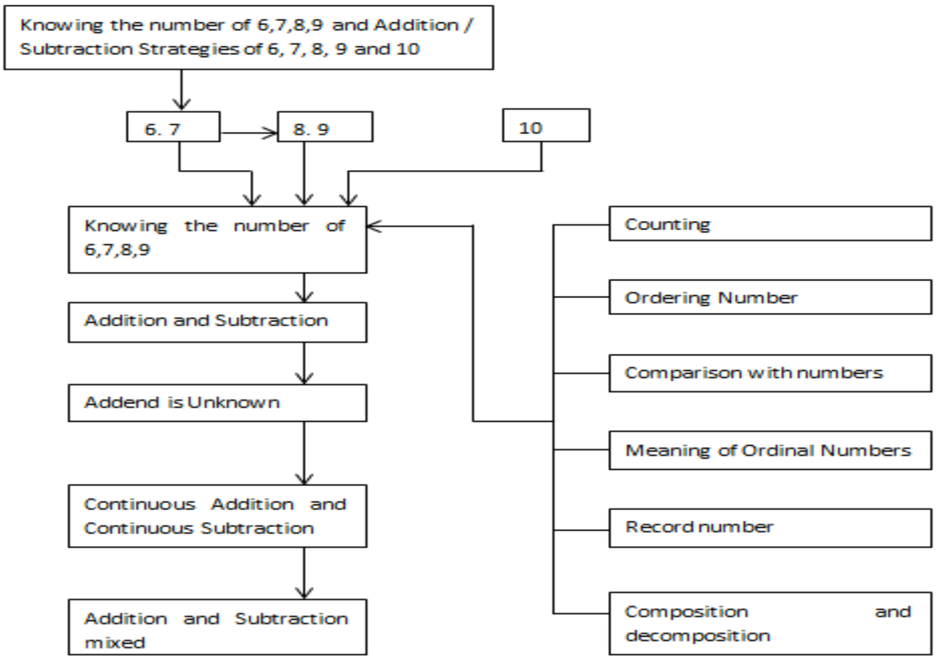

Figure 3.The concept structure of knowing number 6-10 in Chinese teaching reference book (Elementary Mathematic Department, 2005, P.35)

The design above mainly reflect Chinese curriculum tradition with focus on goals – "two bases", namely, the curriculum foundation of addition and subtraction is "part-part-whole" (pre-algebra thinking foundation) relationship. In fact, "two bases" is regarded as the most valuable tradition in the history of Chinese curriculum reform by local experts, different from those in other counties. It is impressive that the concrete foundations, similar to knowledge package (Ma, 1999), in every unit clearly are presented in Chinese teacher guide book. Fig.3 is the concept structure of knowing number 6-10, the concrete curriculum foundation in a unit, in Chinese teaching reference book (Elementary Mathematic Department, 2005, P.35).

Invariant concept vs. variant concepts embedded in Chinese / Portugal textbook examples

Addition and subtraction are almost always connected together using the ‘transformations’ principle of OPMC in Chinese textbooks, rather than separated in different chapters as they are in the Portuguese textbook. It is interesting to note that the Chinese textbook authors did not separate the subtraction concept from the addition concept even in first lesson (Sun, 2011b).

The introduction of addition and subtraction in the first grade by problems with variation

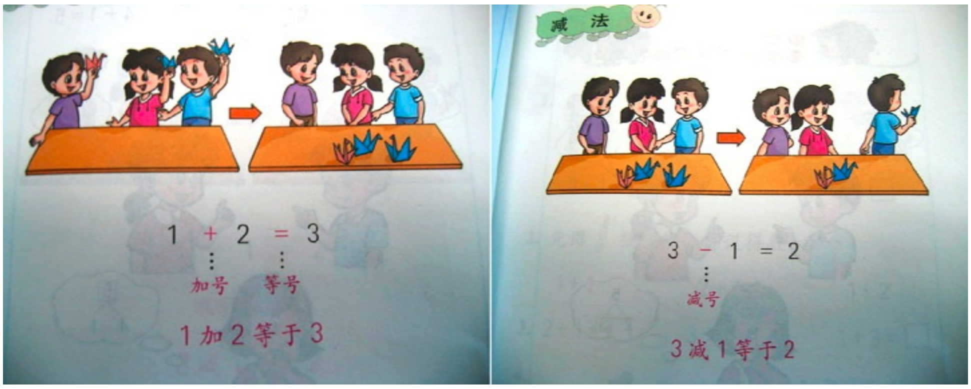

Fig 1. Mathematics Textbook Developer Group for Elementary School, 2005, vol. 1

Figure 1 shows a paradigmatic example of OPMC: xiao ming folds a pink paper crane; xiao li and xiao hua fold two blue paper cranes. How many paper cranes do they fold? The answers are: 1+2=3. There are 3 paper cranes. Xiao ming takes a paper crane. How many paper cranes does he leave? The answers are: 3−1=2. The drawing intends to help learners to recapitulate the relationship of addition and subtraction, and the meaning of “equal” from the problem set 1+2=3, 3−1=2. The problem sets hinges on exemplifying relationships rather than objects and reflects the mathematical structure underlying the problems in this respect. The addition concept is different from that of subtraction, which belongs to a different category. In this way, these two concepts are combines into one category of part-part-whole.

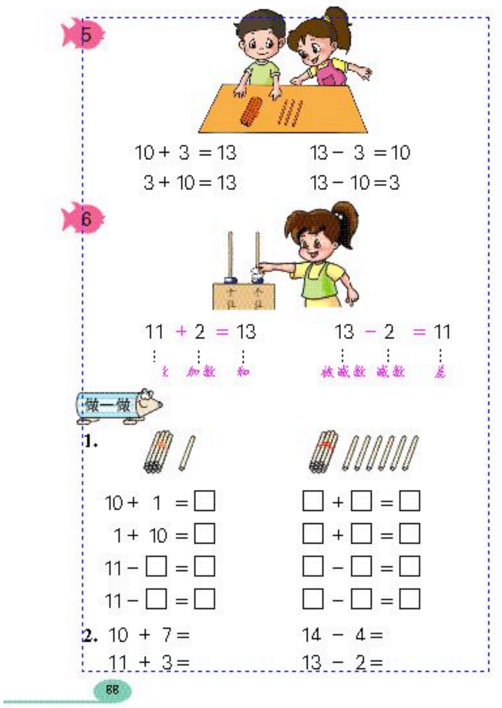

Fig. 2, from Mathematics Textbook Developer Group for Elementary School, 2005, vol. 68

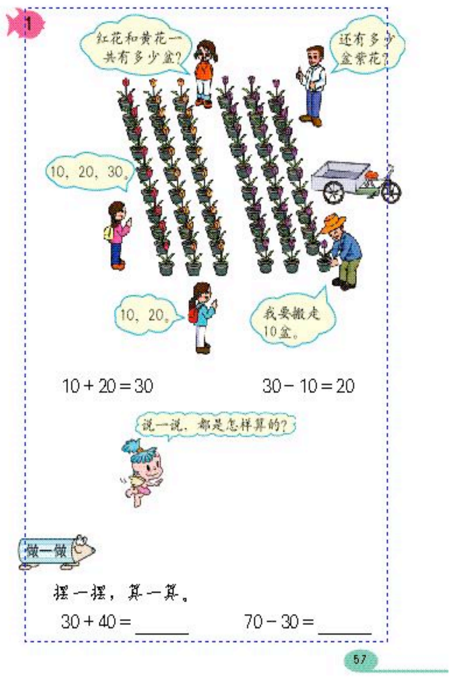

Fig. 3, from Mathematics Textbook Developer Group for Elementary School, 2005, vol. 57

Fig. 2, 3 show two similar paradigmatic examples of problem variation: 10+3=13, 13-3=10, 3+10=13, 13-10=3, and 10+20 = 30; 30-10 = 20.The two problem set intends to help learners to recapitulate the relationship of addition and subtraction, and the meaning of “equal”. The textbook design offers visual models to enable learners to understand the underlying part-part-whole relationship.

The goal and pedagogy of this design is explained below in its reference book.

> The teaching idea of meaning of subtraction is same as that of addition. Textbook use the same situation to elicit subtraction which indicates the relationship that subtraction is the inverse of addition. Therefore, appropriately combining subtraction with addition in teaching will be helpful for students to grasp the relationship and difference of addition and subtraction, which will deepen the understanding of the meaning of addition and subtraction too (Elementary Mathematic Department, 2005, P.39).

Every example in the Portuguese textbook introduced the concept of addition and subtraction without any connection between them. The pages below are from two separate chapters. Addition and subtraction concepts are introduced as counting in the examples below. While the number line model provides a method of calculating the answers, it is unclear whether it enables learners to connect addition and subtraction.

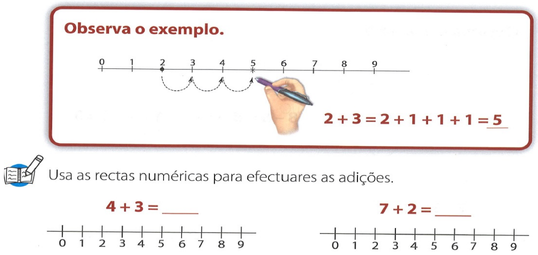

Fig. 4 The example introducing addition concept by counting (Segredos dos Números 1, Matemática 1º ano o Ensino Básico, 2010, p.75)

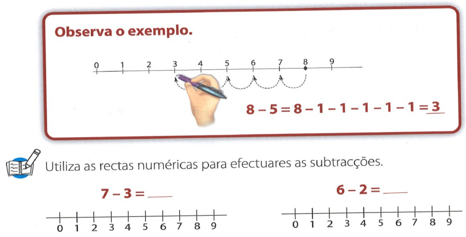

Fig. 5 The example introducing subtraction concept identifying inverse operation (Segredos dos Números 1, Matemática 1º ano o Ensino Básico, 2010, p.76)

Obviously, OPMC play the role by introducing a new concept (i.e., subtraction) from an old concept (i.e., addition) in example 1, to extending the relationship between addition and subtraction into the idea of an equation. Although the Chinese textbook authors appear to use multiple ideas for every example, the underlying invariant concept is of part-part-whole relations and the invariant knowledge is about relations between numbers. In contrast, the addition examples in the Portuguese textbook use multiple underlying concepts, such as, "counting" (as in Figs. 4 and 5), combining and "adding". The subtraction examples in also use multiple concepts: such as "taking away", "comparing", and "identifying inverse operations" but do not connect these simultaneously to the addition concepts. Thus in the Chinese textbooks the meaning of the additive part-part-whole relation is invariant but the way it is represented and enacted varies, but in the Portuguese textbook the meanings of the addition and subtraction concepts vary, and are not connected and it is hard to identify an invariant underlying idea. Thus the space of learning (i.e. what is available to be learnt) is different in the two countries.

Single solution vs. multiple solution methods embedded in every Chinese / Portugal textbook example

In the Chinese textbook multiple related solution methods are almost always elicited together, illustrating the OPMS principle, rather than a single solution method in Portugal textbook.

Fig. 6, An example using variation of connected solution methods (Mathematics Textbook Deveiopers Group for elementary School, 2005, vol.1, p.16)

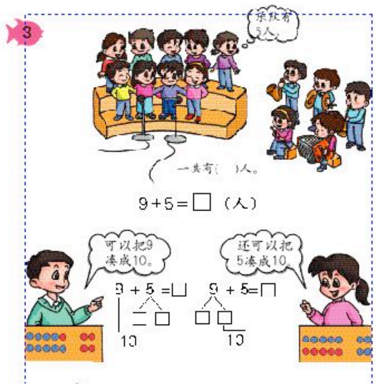

Fig. 7, An example using variation of connected solution methods (Mathematics Textbook Developers Group for elementary School, 2005, vol.1, p.30)

Fig. 6 is a typical example of OPMS in the Chinese textbook. In the problem variation above, 4+1=5 is designed to introduce naturally a solution system of addition. Within the problem set in the example, there are three solution methods given. The first one is that of addition by counting from 1 to 5. The second solution is that of counting from the addend 4 to 5. The third is that of addition by regrouping 5 with 4 and 1. . Obviously, OPMS play the role by weaving all concepts into a complete knowledge tree, introducing a new concept (i.e., regrouping) from an old concept (i.e., counting) in example above, to extending the relationship between three solutions into the idea of an equation. However, these methods are all connected by being about the part-part-whole structure of 5. Fig. 7 is another example of OPMS: 9+5=14. The first solution method is that of addition by regrouping with 5 and 5; the second one is regrouping with 9 and 1, both of which highlight the “make-10 method” and hence the decimal system concept. Obviously, OPMS play the role by weaving a new concept (i.e., decimal system) from old concepts (i.e., regrouping, making-10-concepts) in example above.

The design goal is explained in the following in its reference book.

> Algorithm diversification is one of the basic philosophies of the "new curriculum standard". It states that: "it is natural students use divertive methods because of different living backgrounds and from different perspectives; teachers should respect their thoughts, to encourage them to think independently, to advocate the diversification. (Elementary Mathematic Department, 2005, P.34)

The design pedagogy is explained as follows in Chinese reference book.

> After students` presentation of multiple solutions, teachers may prompt a discussion on which solution is the simplest one, which help them realize the decomposing-solution is simpler than others. Teacher should guide student from the

solution of low level to that of high level. (Elementary Mathematic Department, 2005, P.44)

Compared with Chinese multiple-solution method approach, every example in the Portuguese textbook is intended to be carried out using a single solution method without necessarily connecting it to other approaches in the book. For example, below is a typical design to introduce addition by the specific single solution method of "doubles" or "doubles plus 1"(Fig. 8). The concept on each of these two pages does not vary but the numerical examples do, so, according to the theory, the variation will draw attention to the doubling, or doubling plus one, as the authors intend. However, a learner might not connect this to other models of addition, nor to subtraction, nor be able to choose when to use these methods. This is because there is no invariant concept for addition in the book. To know when to use this method, variation theory suggests that they need to be juxtaposed with some that are not solvable using *"doubles"* or *"doubles plus 1"*. The patterns of variation and invariance from which the learner might discern the underlying conceptual relations are unclear.

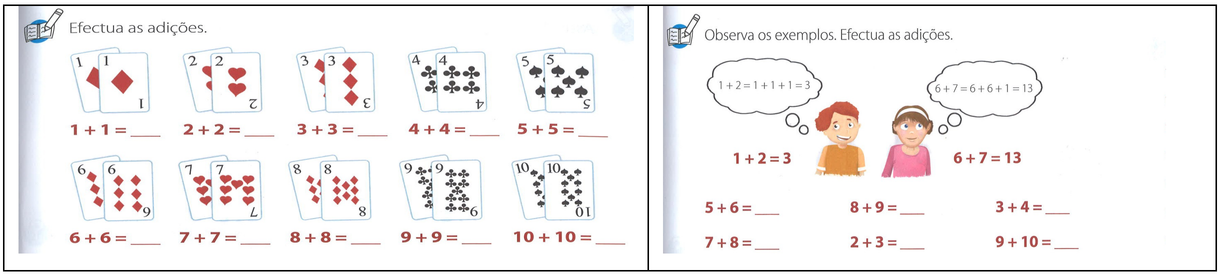

Fig. 8 The example elicits addition by the solution method of *"doubles"* or *"doubles plus 1"* in one of the Portugal textbooks (Segredos dos Números 1, Matemática 1º ano o Ensino Básico, 2010, p.99)

Invariant solution method vs. variant solution methods embedded in Chinese / Portugal textbook examples

Obviously, OPMS play the role by weaving all concepts into a complete knowledge tree, introducing new concepts (i.e., regrouping, decimal system) from old concepts (i.e., counting). Although Chinese textbook authors use multiple solution methods in every example, the particular methods in the Portuguese textbook which depend on counting and doubling are rarely introduced. Only one specific solution method, "make-10"," is addressed explicitly among all the addition /subtraction examples of the first 6 chapters. By contrast, the addition examples in the Portuguese textbook suggest multiple solution methods, such as "*doubles*", "*doubles plus 1*", "*compensation*", (e.g. 6+8=7+7=14) and "*reference number*" (e.g.

6+7=5+1+5+2=10+3=13). The subtraction examples use multiple solution methods such as "*counting back"*, "*the use of tables for the addition to subtraction*", and "*identifying inverse operation of subtraction as addition*" as "basic arithmetic facts" such as, "5+7=12" or "12–7=5") for students simply to memorize. Thus the learner might get a temporary sense of these methods from being offered a variety of suitable examples, but might not get understanding of the whole addition /subtraction relation (e.g., Thevenot, 2010; Savard, Polotskaia, Freiman , 2013).

Chinese / Portugal curriculum goals and pedagogies

Based on the analysis above, we could infer the underlying design principle of 'two bases' by noticing: the 'basic knowledge' foundation of knowing numbers; the 'basic knowledge' foundation of the connection between addition and subtraction as a part-part-whole relation; and the 'basic skill' foundation of the 'make-10 method'. The design principle of 'two bases' obviously different from the Portugal one with focuses on addition, presenting it as combining, and performing it using counting, doubling, compensation, or using and adapting known facts, with each method using its own varied examples and no connection across the textbook. Subtraction is also based on a sequence of concepts, rather than one invariant concept related to addition. The underlying design principle might be 'one thing at a time' broadly used in most textbook development in Europe and throughout the world (Rowland, 2008) and is hence more fragmented and less dependent on laying down basic foundational principles for future work.

This finds is consistent with the operational paradigm and relational paradigm proposed in the topic of addition in the study (Savard , Polotskaia , Freiman , 2013; Sun, 2013c) and in the topic of fraction division (Sun, 2011a). The operational paradigm stress addition and subtraction operations. Relational Paradigm emphasizes addition and subtraction relations, two elements determine a unique third element as a function (Davydov, 1982). Chinese / Portugal curriculum follow relational paradigm and operational paradigm respectively.

This design indicates a relation-orientated idea of task design from the Chinese belief relating to Chinese curriculum development, problem variations play two important roles rooted in Chinese beliefs: introducing new concepts from old concepts (溫故知新) and abstracting invariant concepts from a varied situation and applying these invariant concepts to the varied situations (in Chinese變中發現不變, 以不變應萬變).The design treated three central concepts : knowing number ,addition, subtraction from a holistic view. The design repeats five times from 1 to 5; 6 to 10; 11to 20; 100 to 1000; to 1000 to 10000 ,which aims to bring the invariant concepts into one`s focal awareness" (Marton and Booth , 2007). It tends to be oriented towards repetition, which might be related to the notion of rote learning and memorization (Marton, Watkins & Tang, 1997).But it actually indicated an important Chinese idea of task design：deepening undemanding / making coherence by repetition, which is rarely emphasized in the West. It might reflect a Chinese learning belief, ‘read a hundred times and the meaning will appear ...（书读百遍其义自见）’. However, it is often positioned as the opposite of deep learning and understanding in Western culture (Marton & Saljo, 1976). Watkins & Biggs (2001) further argue that those Western educators who reject rote and repetitive learning may have failed to understand the learning strategy in the Chinese context because the mechanism of repetition is an important part to facilitate memorization and understanding in China.

A “INDIGENOUS” MODEL BASED ON PROBLEM VARIATION PRACTICE: SPIRAL VARIATION CURRICULUM DESIGN MODEL

Obviously, OPMC and OPMS play the role by weaving new concepts from old concepts and stressing the invariant relationship between addition and subtraction, and between counting and regrouping. It is not surprising that the Chinese textbook shows an "indigenous" relation-oriented advantage through variation approach that is absent from the Portugal texts above and American texts (Sun, 2013c). This findings might indicate the difference of curriculum design from “deductive” tradition and “inductive” tradition proposed , expressed as “dialectic mathematics” tradition and “algorithmic mathematics” tradition in the study (Siu, 2009) or operational paradigm and relational paradigm proposed in the study (Savard , Polotskaia , Freiman , 2013). To enable us to see which parts of the different educational systems can learn from each other, it is necessary to point out the underlying meaning beyond the practice from a relationship-oriented inductive tradition, not a non-relationship-oriented deductive one . The underlying meaning could help us better understand the rationales underlying task design by problem variations. Based on the analysis above, problem variations play two important roles rooted in Chinese beliefs: introducing new concepts from old concepts (溫故知新); “no clarification , no comparison” (沒有比較就沒有鑒别), rather than “to consolidate one topic, or skill, before moving on to another,” highlighting invariant by variation and applying invariant to variant situation（以不變應萬變）. These are the rudimentary yet powerful basis .The invariant elements are regarded as intended curriculum core/goal: two bases.

The problem-variation-based reasoning could be formalized for purposes of describing inductive reasoning within an example set as a four-step process, which aims to deepen and develop understanding by circulating, reciprocating process. This view is related to prototype theory, which is most deeply explored in cognitive science.

1. Retrieve: Given a target varied problem, retrieve from prototype examples. A prototype example consists of a problem, its solution, and, typically, annotations about how the solution was derived. For example, the first problem-variation set 1+2=3, and 3−1=2 is a prototype example with concepts of addition and subtraction.

2. Reuse: Map the old solution from the previous prototype example to the target problem. This may involve adapting the additive solution as needed to fit the new varied problem 3−1=2.

3. Revise: Having mapped the previous solution to the target problem, test the new solution in the new problem and, if necessary, revise. In the example above, a student must adapt his/her retrieved addition solutions to include the addition of subtraction concepts by connecting old concept of addition ,sum, and "equal to" to solve the varied problem 3−1=?

4. Retain: After the solution has been successfully adapted to the target problem, store the resulting experience as a new example in memory. In the example above, a student must re-construct new subtraction concepts in the process of adapting his/her retrieved addition solutions. A new conceptual structure including the addition / subtraction concept can be stored as a new mathematics structure and new addition / subtraction example in memory.

Fundamental ingredients of the task design above are the three concepts of prototype examples, recursion arrangement, and differentiation design. As described "because some features of problems are invariant while others are changing, learners are able to see the general through the particular, to generalize, and to experience the particular "in the study (Watson and Mason, 2005).

1. The prototype example set: the examples aim to play a role to study or display concepts and principles as a characteristic representative of its class which facilitates the directing of attention appropriately so as to induce generalizations. They have the highest degree of illustrating; indicating a larger class as a working model; for example, the first example set is 1+2=3, and 3−1=2.
2. Recursion arrangement: problem variation that reduce all other variation problems toward the prototype examples. For example, the paradigmatic example of problem variation: 10+3=13, 13-3=10, 3+10=13; 13-10=3, has repeated structures that could be reduced toward the prototype example set. Within each a unite as a design circle, quantity of number extend from 1 to 5; 6 to 10; 11 to 20; 100 to 1000; and 1000 to 10000.The whole design are recursive in a spiral-like procedure with an invariant addition-subtraction as its conceptual core, two base.

3. Generalization/ specialization process by differentiation design：the design starts with prototype problem variations; it aims to develop the ability of generalization by identifying sameness between a retrieved problem set and the target problem and the ability of specialization by identifying difference between a retrieved problem variations and the target problem. The mathematizing process of generalization/ specialization is developed at same time.

In our past and present research, we follow previous studies (See e. g.

Gu, Huang, & Marton, 2004; Marton, 2008; Sun, 2007) in the search for theoretical models of development of mathematical curriculum design based on Chinese local belief and practice. A relation-orienteted model based on the variation practice : Spiral variation curriculum design model (figture 8) was proposed.

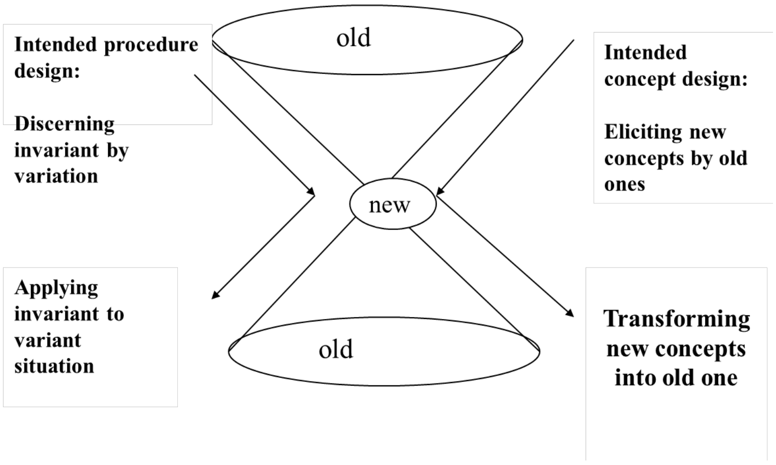

Fig. 8 A model based on the current practice : Spiral variation curriculum design model (Sun, 2007; Sun, 2010; Sun, 2013)

Spiral variation curriculum design model systematically denotes a relationship-oriented model aims to achieve "two bases" by problem variations. The model is situated in the context of teacher-centred teaching styles and textbook centred systems of beliefs. It denotes three important aspects of "variation" in task design for development the hypothetical learning trajectory (Sun, 2006; Sun, 2007; Sun, 2013).

1. Repetition (horizontal aspect of task design) is important for developing familiarity, which lead to knowledge consolidation. Marton, Wen and Wong (2005) argued that the likelihood of being able to recall something was higher if the learners hear or see something several times than if they do not.
2. Variation (vertical aspect of task design) is key for developing learning with a new light , which provide a chance to link new concepts to old concepts .The issue of variations in problem sets directly reflects the old Chinese proverb, "no clarification , no comparison" (沒有比較就沒有鑒別), rather than "to consolidate one topic, or skill, before moving on to another"
3. Invariant (central aspect of task design) is important for developing two base (making coherent learning). Conversely, the invariant concept /solution are not stressed after multiple concepts/solutions are presented in Portugal textbook, which possibly link to fragmentation understanding pointed out by (Ma, 1999).- the core aspect of task design
4. The horizontal , vertical, and central aspect combine intended "spiral" structure (The similar principle of physics states spiral movement can be decomposed into horizontal movement, vertical movement, the centripetal movement)

These principles have been illustrated by the topic of absolute value (Liu, 2004) the topic of division and multiplication of fractions (Sun, 2011b), and the topic of division of fractions (Sun, 2007; Sun, 2011a). A number of studies on its theoretic significance were carried out, including the possible significance of mathematics developed from variation practice (Sun, Wong, & Lam, 2007), possible curriculum significance (Sun, Wong, & Lam, 2006), and its possible characterizations in Chinese mathematics education (Sun, Wong, & Lam, 2005).The model above developed an “indigenous” analytical framework from the “indigenous” interpretation to classify problem organization (Sun, 2011a). In a deeper way, this framework might enable us to see an “indigenous” strategy of making connections of concepts and solutions and enhancing two bases. In the spirit of the Chinese tradition, this strategy brings out the mathematical structure and the rationales behind algorithms, which enrich the development of curriculum from the relation-oriented inductive tradition, rather than the non-relation-oriented deductive tradition in West. In 2006, an exemplary experiment was carried out on a treatment group, where a textbook developed with heavy emphasis on relationships with problem variation in fraction division was used in three schools. In the control group, the traditional HK textbook heavily influenced by England principles and having light emphasis on relationships was used in another three schools. The experimental treatment group achieved a better conceptual understanding of fractions, division, and multiplication compared with the control group (Sun, 2007). Similar experiments on other content areas (ratio, volume, and column) indicated that the findings mentioned above stand [e.g. (Wong, Lam, Sun, & Chan, 2009)]. Due to limited space, I will not extend here. A systematic theoretical introduction could be found in (Sun, 2013b).

THE DEVELOPMENT

Leung (2001) argued that different practices are based on different deep-rooted cultural values and paradigms, whether explicit or implicit. In considering adopting the practices from a different cultural tradition, one has to adapt to local culture and belief. Watson and Mason (2005, 2006) applied variation theory to the field of mathematics education and advice “student-generated examples “ as a tool to promote engagement for conceptual development and further pointed out that the two important parameters of mathematics structure, the dimensions of possible variation and the associated ranges of permissible change, should be emphasized in example usage. The studies (Bartolini Bussi et al., 2011; Bartolini, Sun, Alessandro, 2013) re-designed the Chinese variation task to tailor it to the Italian tradition and to their individual teaching styles and systems of beliefs, in which the task (originally developed in China within a teacher centered and textbook centered tradition) was modified to fit a dialogic approach where teaching and learning are considered the two sides of the same collaborative process. The studies (Sun, 2007; Wong, Lam, Sun, & Chan , 2009) re-designed the variation task to tailor it to the HK context. The use of inductive variation begins with a real life context in which the established concepts are carefully embedded and unfolded in a set of problems that leads to the new concept to be established (the deepening biànshì); and using application biànshì provides new contexts (even created by students) for students to connect (or apply) different acquired concepts (the widening biànshì). The studies (Lo and Marton, 2012; Lo, Pong, & Chik, 2005) re-designed the variation theory to tailor it to the field of teacher education and developed a teacher professional development tool (stressing student variation, teacher variation, and content variation) to guide principle of pedagogical design.

This typical example provides the structures, goals, and pedagogies of variation problems, compared with the Portugal system, in the topics of addition and subtraction in Chinese textbook and its reference book. The comparisons above inspire us to develop much more coherence curriculum by addressing knowledge foundation, concept connections, highlight the invariant concepts and solutions in Chinese mathematics task design system. Variation approaches may be critical in developing concept-connection curriculum and instruction rarely figured out before. It is deserved to note that learning does not necessarily take place solely through learners observing some variant patterns, even if they have generalized them explicitly. Because “learners can do this by focusing on surface syntactic structures rather than deeper mathematical meaning - just following a process with different numbers rather than understanding how the sequence of actions produces an answer” (Watson & Mason, 2006, p.93). However, textbooks in all countries as intended curriculum play important roles in guiding teaching practice and the lesson planning by teachers for both teacher-learning and have a powerful influence on what is learned and how it is learned as a regular “channel” student-learning (Yang, Reys, & Wu, 2010) . Compared with the studies from learning perspective rooted in variation theory used by Marton and his colleagues, from practice to its re-constructed rational explanation, this “indigenous” model is different from any western framework , which takes into account variations as generalizing and specializing conditions in mathematical curriculum design from teaching perspective, which could add a new dimension to variation practice and helps us to understand task design in a holistic view.

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