Lecture about Jean Bourgain [1]

Jean Bourgain *wurde für seine außerordentlichen Beiträge zu mehreren zentralen Gebieten der Analysis geehrt. Dazu gehören die Geometrie der Banachräume, Konvexität in hochdimensionalen Räumen, harmonische Analysis, Ergodentheorie und die Theorie der nichtlinearen Evolutionsgleichungen. Er hat diese Gebiete nicht nur stimuliert, indem er eine Reihe von Problemen löste, die für lange Zeit offen standen, sondern auch, indem er neue Wege zu Fragen aufzeigte, bei denen Fortschritte seit vielen Jahren blockiert schienen. Er gelangte zu seinen Ergebnissen durch die Entwicklung neuartiger Techniken und indem er auf bislang unbekannte Weise verschiedene andere Bereiche der Mathematik, wie Zahlentheorie, Kombinatorik und Stochastik, ins Spiel brachte. Er entdeckte delikate Beziehungen zwischen Geometrie und Fourier-Analysis und hat so fundamentale Entwicklungen in der reellen Analysis und der Theorie partieller Differentialgleichungen eingeleitet. Seine Beiträge werden die Analysis der Zukunft für lange Zeit prägen.*
Auszug aus der Laudation von **Luis A. Caffarelli**.

Let me try to give a flavour of his work through a few remarks on selected aspects of his work. This seletion is of course very limited and strongly influenced by my taste and knowledge. I thus apologize, both to him and his coolaborators for this choice being poor and incomplete.

From his contributions to **functional analysis**, let me just mention his solution, in collaboration with Milman of Santalo's conjecture: Let K and K^* be dual unit balls for some norm, then

$$\mathrm{Vol}(K)\,\mathrm{Vol}(K^*) \geq c^n\,\mathrm{vol}(B_n)^2$$

This result depends in a delicate, sharp refinement, by Bourgain, of Dvoretzky's Theorem.

Let me jump now to **harmonic analysis**. Here Bourgain contributions include:

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1. The boundedness of the spherical maximal function

$$Mf(x) = \sup_r \frac{1}{A(S_r)} \int f(y-x)\, dA(y)$$

($A(S_r)$ the area of the sphere of radius r) in $L^p(R^2)$, $p > 2$ closing the gap of Stein's theory.

2. The boundedness of the "solid" maximal function

$$\overline{M}f(x) = \sup_r \frac{1}{\text{vol}(tK)} \int f(y-x)\, dy$$

in $L^p(R^n)$ with bounds independent of dimension n. (Generalizing Stein's theorem for balls.)

3. The fact that, for any domain Ω, harmonic measure in R^n is supported on a set of Hausdorff measure $n - \varepsilon(n)$.

4. His results on the restriction of Fourier transform to R^n.

Ergodic theory. The work of Bourgain in ergodic theory deals with rather subtle problems associated with pointwise convergence. One may wonder whether the questions asked are too specialized, since the companion information about convergence in mean is much easier to prove and quite often sufficient for applications. On the other hand, pointwise convergence is not an artificial notion, and maximal inequalities, the tool used to prove pointwise convergence, appear so often in harmonic analysis that there is no question that they are of fundamental importance.

Let (X, μ, T) be a dynamical system where T is a measure preserving transformation of X. The celebration ergodic theorem of Birkhoff asserts that if $f \in L^1(X)$ then

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x) \to \int_X f(x)\, d\mu(x)$$

almost surely for $x \in X$. Some generalisations were known, but none replacing n by a "thin" sequence k_n. Convergence in mean, in the case $f \in L^2$, for a thin sequence like n^2 was known from the work of Bellow and Fürstenberg, but the question remained about the generalisation of Birkhoff's theorem to thin sequences. In 1987, Bourgain answered in the affirmative the question, showing that if $f \in L^2(T)$ is periodic with period 1 and $p(n)$ is a polynomial with integer coefficients then

$$\frac{1}{N}\sum_{n=1}^{N} f(T^{p(n)}x) \to \int_0^1 f(x)\,dx$$

almost surely for $x \in T$. The method of proof is completely new, and is a real *tour de force*. Bourgain quickly reduces the problem to proving a maximal inequality for functions in $L^2(Z)$: If Mf is the maximal operator

$$Mf(k) = \sup_N \left| \frac{1}{N}\sum_{n=1}^{N} f(k+p(n)) \right|,$$

then $||Mf||_2 \leq C||f||_2$. Bourgain's entirely new idea in this context is to pass to the Fourier transform on the shift by $p(n)$, and then use a technique inspired by the Hardy and Littlewood circle method to bring home the result. This is very hard to do and, besides the Hardy and Littlewood division into major and minor arcs, requires the theory of martingales in probability theory, the Poisson semigroup, several hard maximal inequalities and estimates of exponential sums.

More recently, Bourgain was able to establish, with $f \in L^2(T)$ periodic with period 1, the almost sure convergence of

$$\frac{1}{N}\sum_{n=1}^{N} f(\lambda^n x) \to \int_0^1 f(x)\,dx$$

where $\lambda > 1$ is *algebraic*. Again, he reduces the problem to a maximal inequality, but the proof is far from being routine. Even if $\lambda = 3/2$, the result is new. In this case, nothing is known about the associated exponential sum, so that he has to introduce new number theoretic techniques to get around this difficulty.

Non-linear partial differential equations. Bourgain's work in this field deals with dispersive equations in the periodic setting. Here he has obtained non-trivial well-posedness results for singular initial data, combining ideas from harmonic analysis, number theory together with the introduction of some new non-linear funtion spaces in which the non-linear iteration is carried out. The program is remarkable an dvery general, and these ideas are bound to have many more applications.

The first result, dating to 1992, deals with a non-linear Schrödinger equation

$$iu_t + \Delta u + u|u|^\alpha = 0$$

with Cauchy data at time $t = 0$. In R^n this problem is well-posed locally in time in the Sobolev space of initial data $H^s(R^n)$, provided $s > n/2$. Similar results were known in the periodic case T^n. However, in many problems in statistical mechanics one needs to start with much less regular data, for instance Brownian paths, and the standard method using the so-called Strichartz norm estimates fails completely. Bourgain showed that, in the periodic setting, the missing Strichartz norm estimates can be replaced by a weaker version, which can be established by means of analytic number theory techniques, provided one performs the crucial iteration not in mixed norm spaces, but in new function spaces defined through conditions on the Fourier transform. As an example, if $n = 1$ this yields well-posedness in $L^2(T)$ for $0 < \alpha \leq 2$.

Even more striking are his results on the well-posedness of the initial value problem for the Korteweg-De Vries equation

$$u_t + u_{xxx} + uu_x = 0$$

associated to solitary waves in a channel. In 1983, Kato showed that there are weak solutions in $L^2(R)$ or in $H^1(R)$, but the uniqueness of these solutions remained an outstanding problem. In 1991, Kenig-Ponce-Vega showed well-posedness in $H^s(R)$, $s > 3/4$, settling the uniqueness question in $H^1(R)$. however in the periodic case T, nothing better than well-posedness in $H^s(T)$, $s > 3/2$ was known and the uniqueness problem was still open. Using his new method, Bourgain showed well-posedness in $L^2(T)$, thus solving the uniqueness problem. Moreover, by combining ideas of McKean und Trubowitz with his new method, he went on to show that L^2 solutions are quasi-periodic in time, solving a conjecture of Lax, and extended his method to the non-periodic case, showing well-posedness in $L^2(R)$ and establishing the uniqueness of Kato's solutions. All this is a major step forward in the problem.