Proceedings of the International Congress of Mathematicians
Hyderabad, India, 2010

Work of Louis Nirenberg

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Louis Nirenberg is one of the most outstanding analysts of the twentieth century. For more than half a century, he has been a world leader in partial differential equations – a master of inequalities and regularity theory – with fundamental contribution in geometry, complex analysis, and fluid dynamics. Nirenberg is a household name in these fields. In addition to the depth and its basic importance, his work also has enormous influence on others. In each of the last 10 years, top 15 cited papers in mathematics include at least 2 of Nirenberg′s, according to the MathSciNet. Working with others has been an essential part of Nirenberg′s research — more than 90% of his research are joint works.

1. Geometry

In 1953 Nirenberg published 4 papers. The first two papers ([1] and [2]) solved two long standing open problems, the Weyl problem and the Minkowski problem, in differential geometry, and, in partial differential equations, gave basic estimates to solutions of nonlinear second order elliptic equations in dimension two. His solution of the Weyl and Minkowski problem was a pioneering work in the study of geometry problems using nonlinear partial differential equations, and a milestone in global geometry. These were established in his Ph.D. thesis in 1949. He was slow in rewriting the thesis for publication. His thesis adviser was J.J. Stoker, himself a student of Heinz Hopf.

The Weyl problem, raised by H. Weyl in 1916, is the following: Given a smooth metric g of positive Gauss curvature on the sphere S^2, is there an embedding $X : S^2 \to R^3$ such that the metric induced on S^2 by this embedding is g? Such an embedding $(S^2, g) \to R^3$ is called isometric, and satisfies the following system of nonlinear partial differential equations:

$$\nabla_i X \cdot \nabla_j X = g_{ij}.$$

Such an isometric embedding, if it exists, is unique up to rigid motion. H. Lewy proved in 1938 the existence part under the assumption that the metric g is analytic, using theorems he developed concerning analytic Monge-Ampère equations. Nirenberg gave a beautiful solution of the Weyl problem, using the method of continuity and the strong apriori estimates he established for nonlinear elliptic equations in two dimension. The Weyl problem was independently solved by A.V. Pogorelov using a different method.

Given a closed smooth strictly convex surface M in the Euclidean space R^3, the Gauss map $\nu : M \to S^2$, mapping a point P on M to the unit outer normal of M at P, is a diffeomorphism. The Gauss curvature K_M of M, identified as a function on S^2 through ν, satisfies

$$\int_{S^2} \frac{x}{K_M(\nu^{-1}(x))} = 0,$$

where $x = (x_1, x_2, x_3)$ is the coordinate function on S^2.

The Minkowski problem concerns the converse: Given a smooth positive function K on S^2 satisfying

$$\int_{S^2} \frac{x}{K(x)} = 0,$$

is there a closed smooth strictly convex surface M in R^3 whose Gauss curvature K_M is given by $K_M(P) = K(\nu(P))$? Nirenberg gave an affirmative answer to the question.

Consider a nonlinear partial differential equation of second order

$$F(x, u, \nabla u, \nabla^2 u) = 0 \quad \text{in } B_1,$$

where B_1 is a unit ball of R^n.

We assume that F is uniformly elliptic: F is a smooth function of its arguments satisfying, for some positive constant Λ,

$$\frac{1}{\Lambda}|\xi|^2 \leq \frac{\partial F}{\partial M_{ij}}(x, s, p, M)\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad \forall\ \xi \in R^n,\ \forall\ (x, s, p, M).$$

A problem of basic importance is whether there exists a bound of some Hölder norm of the Hessian $\nabla^2 u$ in half of the ball in terms of the L^∞ norms of $|u|$, $|\nabla u|$ and $|\nabla^2 u|$ in B_1. Nirenberg established such an estimate in dimension $n = 2$ as mentioned above. On the other hand, such an estimate does not hold in dimension $n \geq 12$, as shown recently by N. Nadirashvili and S. Vladut, while the problem remains open in dimension $3 \leq n \leq 11$.

With P. Hartman [11], Nirenberg studied spherical image maps where Jacobian do not change sign. In particular it was shown that if u is a real function on R^2 with the determinant of its Hessian equal to zero, then its graph is a cylinder.

Nirenberg proved in [13] rigidity of a class of surfaces in R^3 which have handles. There is still the open problem of whether a smooth closed surface in R^3 can be deformed continuously, in an isometric way. Namely, as E. Calabi expressed it, is there Nature′s accordion?

In a work with C. Loewner [24], Nirenberg solved a nonlinear problem coming from geometry which involves treating some nonlinear partial differential equations invariant under conformal or projective transformations and finding solutions which become infinite on the boundary.

There is a famous result of A.D. Alexandrov that a connected compact smooth hypersurface embedded in the Euclidean space with constant mean curvature is a sphere. With Y.Y. Li, Nirenberg proved in [57], a paper dedicated to the memory of S.S. Chern, a generalization of this, replacing the constancy by a monotonicity condition, and proving symmetry of the hypersurface about a hyperplane under some conditions. Open questions still remain, in particular, about possible extension of Hopf Lemma for elliptic operators. They also studied in [56] the regularity of the distance function to the boundary, in Finsler geometry, and applied it to the study of singular set of viscosity solutions of Hamilton-Jacobi equations.

2. Linear Partial Differential Equations.

One of the 4 papers Nirenberg wrote in 1953 is [3] in which he proved the strong maximum principle for parabolic operators extending the classical results for elliptic operators. This has been a standard reference.

In 1956 Nirenberg settled in [4] a long standing open problem about regularity of elliptic boundary value problems up to the boundary. He did this for equations of arbitrary order.

Much of Nirenberg's work concerns estimates for solutions of elliptic boundary value problems. With S. Agmon and A. Douglis he gave in [10, 16] a comprehensive treatment of linear elliptic partial differential equations of any order with general boundary conditions which include the extension of Schauder and L^p theory for second order elliptic partial differential equations with Dirichlet boundary condition to this generality. These fundamental results are used every day by researchers in partial differential equations, fluid dynamics, material sciences, and many other fields.

With C.B. Morrey, Nirenberg proved in [6] the analyticity of solutions of general linear elliptic systems with analytic coefficients.

In [15], a long paper with Agmon, Nirenberg investigated solutions of ordinary differential equations in Banach space with applications to asymptotic expansion as $t \to \infty$ for solutions of elliptic equations in a cylinder. These results have been used by others as the basis for the study of elliptic equations in domains with corners: behavior of the solutions near the corners. The paper with Agmon was later generalized by A. Pazy to consider equations with coefficients depending on t. Pazy's result was used much later by H. Berestycki and Nirenberg in their study of traveling fronts in cylinder in flame propagation problems.

A basic problem in complex analysis is the so-called $\bar{\partial}$ Neumann problem. It was solved by J.J. Kohn. Nirenberg and Kohn then extended the regularity result to a wide class of noncoersive boundary value problems. This involves the loss of some derivatives. In order to do this they found it necessary to extend the Calderon-Zygmund theory of singular integral operators, to make an algebra of such operators. They introduced in [17], in 1965, the theory of pseudo-differential operators. This is a basic tool which has led to much further development in microlocal analysis and in partial differential equations. They also studied in [18] degenerate elliptic-parabolic equations.

There was a famous example by Hans Lewy of an operator of the form

$$\sum_{j=1}^{3} a_j \frac{\partial u}{\partial x_j} = f \quad \text{in } R^3 \tag{1}$$

with complex coefficients $\{a_j(x)\}$, showing that there is no complex solution. The construction was motivated by complex analysis in C^2. With Trèves [14], Nirenberg discovered the general condition under which the equation is solvable. In later work [21] and [22] they treated general linear partial differential operators and formulated a condition Φ for local solvability. For pseudo differential operators they introduced a more general condition Ψ. They proved sufficiency of Ψ in the analytic case. Sufficiency of Φ for partial differential operators was proved by R. Beals and C. Fefferman. Necessity of Ψ was proved by R.D. Moyer in dimension two and by L. Hörmander in general. Only a few years ago was sufficiency of Ψ proved by N. Dencker.

Later, Nirenberg considered again in [25] the equation (1) with $f = 0$, and constructed an operator L for which the only solution of $Lu = 0$ is u =constant. This gave a surprising solution to a problem of Hans Lewy. This is connected with complex analysis in C^2. The corresponding question in C^n, or in R^{2n+1}, was taken up by M. Kuranishi who proved that for $n \geq 3$ the system had solutions. The case $n = 2$ is still open.

With H. Berestycki and S.R.S. Varadhan [49], Nirenberg proved the existence of principal eigenvalue (necessarily real), and derived improved form of the maximum principle for general linear second order elliptic operators in general domains. The work is considered a classic. Many people continue to refer to it.

3. Inequalities

Inequalities play a central role in almost all of Nirenberg$'$ work. He proved basic interpolation inequalities and embedding inequalities, which are used every day.

In [9], lectures in partial differential equations, one of the lectures established general interpolation inequalities between functions and their derivatives, involving L^p spaces. These are called the Gagliado-Nirenberg inequalities. Gagliado derived them at the same time. These results are used all the time.

With F. John, Nirenberg introduced in [12] the space of functions of bounded mean oscillation (BMO): those are locally integrable functions f on R^n satisfying, for some constant C,

$$\frac{1}{|B|}\int_B |f(x) - f_B|dx \leq C, \qquad \text{for all balls } B,$$

where $f_B = |B|^{-1}\int_B f$ denotes the mean value of f over the ball. This is a new class of functions between L^p and L^∞. They established a deep basic estimate for BMO functions: If f is in BMO, then u has exponential integrability, i.e., for some positive constant a,

$$\int_B e^{a(f(x)-f_B)}dx < \infty, \qquad \text{for all balls } B.$$

This seminal result has become a central element in analysis and is much used in partial differential equations, harmonic analysis, and probability theory.

With H. Brezis, Nirenberg extended in [50] and [51] degree theory to maps, between manifolds, which are merely in VMO (have vanishing mean oscillation), a refinement due to D. Sarason of the class BMO. The maps need not be continuous. The need for having a degree for such maps, with the useful properties, arose in problems coming from harmonic maps, Ginzburg-Landau equations, among others.

4. Complex Analysis

With A. Newlander, Nirenberg solved in [5] the problem on integrability of almost complex structures. In fact it was suggested to Nirenberg by A. Weil and S.S. Chern. This was a basic problem in higher dimensional complex analysis: When can one reduce a given system of n first order linear partial differential equations in R^{2n} to the Cauchy-Riemann equations in C^n, after a smooth change of coordinates? Necessary conditions were long known. Here their sufficiency was proved. This was extended by Nirenberg in [8] to a complex form of the classical Frobenius Theorem about differential forms. The Newlander-Nirenberg theorem reminds me of a meeting with S.S. Chern some years ago at the Chern Institute in Tianjin during which he gave me an envelope containing a mathematics manuscript and a letter to Nirenberg, and asked me to bring it to Nirenberg. He told me that the envelope was not sealed in case I wanted to make a copy, which I did. I remember that his whole manuscript had only two references, one of them is the paper [5].

With K. Kodaira and D. Spencer, Nirenberg proved in [7] the existence of deformation of complex structure on complex manifolds. The more general form of this result was later obtained by M. Kuranishi.

Nirenberg wrote one paper [19] with S.S. Chern (together with H.I. Levine). It introduced intrinsic norms on the homology groups of a complex manifold.

In [20] Nirenberg gave a rather simple proof of the Malgrange extension of the Weisstrass preparation theorem.

In [23] he proved an abstract form of the Cauchy-Kowalewski Theorem. This was later improved by T. Nishida, and applied to fluid dynamics.

With Caffarelli, J.J. Kohn and Spruck, Nirenberg solved in [36] the Dirichlet problem for degenerate complex Monge-Ampère equations and some uniformly elliptic ones.

5. Nonlinear Partial Differential Equations and Applications

With D. Kinderlehrer and J. Spruck, Nirenberg wrote a series of papers ([27]-[29]) on regularity of free boundaries, in the obstacle and other problems, including generalization of a result of Hans Lewy. With H. Berestycki and L. Caffarelli he wrote a deep paper [42] on uniform estimates for regularization of free boundary problem.

With B. Gidas and W.-M. Ni, Nirenberg wrote two papers [30, 32] on symmetry and monotonicity of positive solutions of various second order elliptic problems. They used the method of moving planes, due originally to A.D. Alexandrov and then used by J. Serrin. Since then, this method has found surprising applications to a wide variety of problems including derivation of a priori estimates. Later with H. Berestycki, Nirenberg gave in [45] a significant modification to the argument so that it applies to domains whose boundary maybe irregular, and also introduced the sliding method to prove monotonicity.

In all these, the maximum principle plays a crucial role. Many of Nirenberg′s papers rely on the maximum principle, in one form or another. As he said, with his ever-present sense of humor, “I have made a living from the maximum principle”.

Nirenberg wrote a paper with L. Caffarelli and R. Kohn in fluid dynamics [33] in 1982, on incompressible Navier-Stokes equation in three space dimensions. The equations describe the motion of an incompressible fluid in R^3 (or a domain of it), which are satisfied by unknown velocity function $u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ and pressure function $p(x,t)$, defined for position $x \in R^3$ and time $t \geq 0$. They take the form (with zero external force and unit viscosity — for simplicity)

$$\frac{\partial u^i}{\partial t} + \sum_{j=1}^{3} u^j \frac{\partial u^i}{\partial x_j} - \Delta u^i + \frac{\partial p}{\partial x_i} = 0, \quad x \in R^3, t \geq 0,$$

$$\sum_{i=1}^{3} \frac{\partial u^i}{\partial x_i} = 0, \quad x \in R^3, t \geq 0,$$

and

$$u(x,0) = u_0(x),$$

where $u_0(x)$ is a given smooth divergence-free vector valued function with compact support.

In order to find smooth solutions u and p, one tries to first establish the existence of solutions in weak sense and then prove regularity of weak solutions. For physically reasonable solutions, $|u|^2$ should satisfy suitable growth property.

J. Leray proved in 1934 the existence of weak solutions with suitable growth property. Nirenberg, in the joint work with Caffarelli and Kohn, proved that the $1-$dimensional Hausdorff measure of the singular set of physically reasonable weak solutions, if singularities arise, is zero (so it can not contain a curve in space-time). Up to now this result has not been improved. The question of whether singularities can occur is a basic open problem in analysis and partial differential equations, and is one of the seven Clay Mathematics Institute Millennium Prize Problems.

With H. Brezis, Nirenberg proved deep existence and nonexistence results in [34] on semi-linear elliptic equations with critical exponent. This work has inspired many researchers working on problems with lack of compactness, and has led to much research activity in calculus of variations and in partial differential equations. The paper is referred to constantly. Beside this, they wrote a large number of joint papers, some are on semi-linear equation and critical point theory (e.g. [26], [31], [46] and [47], one with J.M. Coron).

With Caffarelli and Spruck, Nirenberg wrote a series of papers ([35]-[41], one with J.J. Kohn) on fully nonlinear elliptic equations, such as the Monge-Ampere equations and derived new existence theories, some with applications in differential geometry. All these papers involve deep, intricate estimates, and use the maximum principle. These works have led to much outstanding research in fully nonlinear partial differential equations.

Berestycki and Nirenberg wrote a series of papers on traveling fronts in cylinder (e.g. [43] and [44]), and with Caffarelli they wrote several papers ([48], [52], [53]-[54]) on properties of solutions of equations in unbounded domains, using the method of moving planes and the sliding method, and they obtained existence for solutions.

With Y.Y. Li, Nirenberg also wrote a large number of papers on different topics; one [55] involves estimates for elliptic systems (such as equations of elasticity) coming from composite material. With L. Caffarelli and Y.Y. Li, Nirenberg is writing a series of papers ([58]-[60]) on singular solutions of nonlinear elliptic equations.

Nirenberg has shared with mathematicians all over the world his knowledge, his wisdom and his friendship. He transmitted to generations of young mathematicians his love for mathematics, gave them guidance, taught them to think and do research. He has supervised 45 Ph.D. students.

Among his many honors and awards, he received in 1959, the Bôcher Prize of the American Mathematical Society; in 1982, the Crafoord Prize which was established by the Royal Swedish Academy of Sciences in areas not covered by the Nobel Prizes; in 1994, the Steele Prize for Lifetime Achievement of the AMS; and in 1995, the National Medal of Sciences of the United States.

We draw attention to an interview of Nirenberg [Interview with Louis Nirenberg, Notices of the AMS, April 2002], and articles on the works of Nirenberg written by L. Caffarelli and J.J. Kohn [Louis Nirenberg receives National Medal of Science, Notices of the AMS, October 1996].

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