The work of Elon Lindenstrauss

Harry Furstenberg

I've been asked to describe some of the achievements of Elon Lindenstrauss - our Fields medalist. Elon Lindenstrauss's work continues a tradition of interaction between dynamical systems theory and diophantine analysis. This tradition goes back at least to the year 1914 - when Hermann Weyl published a paper entitled "An application of number theory to statistical mechanics and the theory of perturbations." In that paper Weyl used what we would call Kronecker's Theorem to show the validity of the ergodic hypothesis in certain situations. In the meantime the roles have been reversed, with dynamical systems theory and ergodic theory providing the tools for answering questions in number theory.

The number theoretical issues arising in the work of Lindenstrauss have to do with so-called diophantine approximation - in which one asks whether inequalities having real solutions have integer solutions. In this area we encounter a phenomenon which is reminiscent of ergodic behavior. It can be described crudely by saying that whatever is not excluded for some good reason and can happen in principle, will eventually happen - at least approximately. There is a good reason that

$$-\varepsilon < x^2 - (1+\sqrt{2})^2 y^2 < \varepsilon$$

cannot be solved for small ε (this would imply that $\sqrt{2}$ is well approximable). But this doesn't apply to the three variable inequality:

$$-\varepsilon < x^2 - (1+\sqrt{2})^2 y^2 - \alpha z^2 < \varepsilon \quad (\alpha \neq 0 \text{ arbitrary})$$

and indeed by the relatively recently established Oppenheim conjecture, for any positive ε, this has a solution in integers (x, y, z) not all 0.

An important advance has come about by enlarging the scope of dynamics to include what will be referred to as "homogeneous dynamics". Every since Poincaré dynamical theory had broken out of the shackles of Ordinary Differential Equations and a dynamical system comes about whenever we have a 1-parameter group $\{T_t\}$ — think of t as time — of transformations acting in a space X, which we identify as the phase space of the system. We have *homogeneous* dynamics when X is a homogeneous space of a Lie group; we can write $X = G/\Gamma$. For any 1-parameter subgroup $\{g(t)\} \subset G$ we can set $T_t(g\Gamma) = g(t)g\Gamma$. Homogeneous dynamics allows one further abuse of the term "dynamics", extending the action from a 1-parameter subgroup of G to an arbitrary Lie subgroup $H \subset G$, so that the time parameter can be higher dimensional.

This liberalization of viewpoint has been quite fruitful in the recent application of dynamics to number theory.

One particular homogeneous space has been the focus of activity in this work, it is a space that appears implicitly in Minkowski's geometry of numbers. Namely, for a dimension d, we consider the space Ω_d of unimodular lattices spanned by d independent vectors in $\mathbb{R}^d$. The group $\mathrm{SL}(d, \mathbb{R})$ acts transitively on this space in a natural way: $\Omega_d \cong \mathrm{SL(d, \mathbb{R})/SL(d, \mathbb{Z})}$. There is a measure on Ω_d invariant under the action of the group and the measure of Ω_d is finite. Nonetheless the space Ω_d is non-compact in its natural topology. This is important, as is Mahler's criterion for a set $\Sigma \subset \Omega_d$ to have compact closure. Namely, $\bar{\Sigma}$ is compact unless there is a sequence $\{\sigma_n\} \subset \Sigma$ and vectors $v_n \in \sigma_n$ with $\|v_n\| \to 0$.

There is a broad spectrum of problems for which this is relevant. Namely, let $\Phi(x_1, x_2, \ldots, x_d)$ be a homogeneous polynomial and we ask if for arbitrarily small $\varepsilon > 0$ one can solve $|\Phi(x_1, x_2, \ldots, x_1)| < \varepsilon$ in integers not all 0. (This would in fact imply that the range of Φ on $\mathbb{Z}^d$ is dense in either $\mathbb{R}^+, \mathbb{R}^-$, or both). Now define the subgroups

$$H_\Phi \subset G = \mathrm{SL(d, \mathbb{R})} \ \text{ by } \mathrm{H_\Phi = \{h \in G : \Phi(h\bar{v}) = \Phi(\bar{v}) \ \text{ for all } \bar{v} \in \mathbb{R}^d\}}.$$

In general for a non-compact group H, one expects orbits Hx to be unbounded, and then Mahler's criterion will come into play. If we take $x_0 \in \Omega_d$ to be the lattice $\mathbb{Z}^d$, then if $H_\Phi x_0$ is unbounded, this will imply that there exist $h \in H_\Phi$ and $\vec{v} \in \mathbb{Z}^d$ with $\|h\vec{v}\|$ arbitrarily small which means that $\Phi(\vec{v})$ is arbitrarily small. This was the strategy leading to the solution of the Oppenheim conjecture in the 80's by Margulis. Here $\Phi(x_1, x_2, x_3) = \alpha x_1^2 - \beta x_2^2 - \gamma x_3^2$ and H_Φ has the property investigated by Marina Ratner motivated by conjectures of Raghunathan and Dani - of being generated by unipotent subgroups. (A linear transformation is unipotent if 1 is its unique eigenvalue.) By this theory one can classify all the closed H_Φ-invariant subsets of Ω_3 and in particular, one sees that an H_Φ-orbit has compact closure only if it is already compact. Margulis shows that this can happen to the orbit of $x_0 = \mathbb{Z}^d$ only if α, β, γ are commensurable. Otherwise this orbit is unbounded which leads to the conclusion that $|\Phi(x_1, x_2, x_3)| < \varepsilon$ has integer solutions.

Another notorious diophantine approximation problem is Littlewood's conjecture: for *all* pairs of real number α, β, if for x real we denote by $\|x\|$ the distance of x to the nearest integer, then

$$\liminf_{n \to \infty} n\|n\alpha\| \ \|n\beta\| = 0.$$

This fits into the framework just discussed for the polynomial

$$\psi(x_1, x_2, x_3) = x_1(\alpha x_1 - x_2)(\beta x_1 - x_3)$$

where we disallow $x_1 = 0$. A linear transformation carries this to

$$\Theta(X, Y, Z) = XYZ$$

and H_Θ is (locally) just the diagonal subgroup $\left\{ \begin{pmatrix} e^{-t-s} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^s \end{pmatrix} \right\}$. This has no non-trivial unipotent subgroups; and the Ratner theory does not apply. Nonetheless, Margulis has conjectured that a bounded orbit for H_θ is necessarily compact and this conjecture, as in the foregoing discussion, has the Littlewood conjecture as a consequence.

We have here a contrast of unipotent homogeneous dynamics with what might be called — with Katok — higher rank hyperbolic dynamics. The former is "tame": neighboring points separate at a polynomial rate, whereas in hyperbolic dynamics they can separate at an exponential rate. Thanks largely to the work of Ratner, the unipotent theory may be said to be largely understood, whereas the hyperbolic theory is in a less satisfactory shape.

The earliest confirmations of Raghunathan's conjectures for unipotent actions came from the case $d = 2$ with results regarding the horocycle flow which corresponds to the subgroup $\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}$. The hyperbolic counterpart, $\left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\}$, leads to the geodesic flow which is the prototypical example of chaotic dynamics. This would lead one to expect that the higher dimensional cases of diagonal group actions can only get worse, thus leaving little hope for a dynamical approach to the Littlewood conjecture.

Among those who spearheaded the initiative to understand the phenomenon of rigidity in the hyperbolic framework was Anatole Katok, who, in a paper with Ralph Spatzier gave conditions for a rigidity result in the hyperbolic setup. In this paper the importance of the acting group being of rank ≥ 2 is underscored. An analogy is drawn to a phenomenon I have studied; namely the paucity of closed subsets of the group $\mathbb{R}/\mathbb{Z}$ invariant under two endomorphisms $x \to px$ (mod 1)and $x \to qx$ (mod 1), provided $\{p^n q^m\}$ is not contained in some $\{r^n\}$. (That is to say $\log p / \log q$ is irrational). The only closed sets are $\mathbb{R}/\mathbb{Z}$ itself and finite sets of rationals. It is an open question whether the only invariant measures are correspondingly the obvious ones: Lebesgue measure and atomic measures supported on rational and combinations of these. This example has been instructive for the following reason. Namely if one adds the condition that one or the other transformation, $x \to px$ or qx (mod 1) has *positive entropy* with respect to the invariant measure in question, then the measure must have a Lebesgue component. This result of Dan Rudolph which partially answers our query regarding $\times p, \times q$ suggests that for diagonal homogeneous actions, positive entropy will also play a significant role. This is the case already in the paper of Katok and Spatzier where other hypotheses are necessary. The state-of-the-art theorem in this regard is due to Einsiedler, Katok and Lindenstrauss and it depends heavily on new ideas of Lindenstrauss, requiring only positive entropy along some 1-parameter subgroup to conclude that an invariant measure is of an algebraic character. This theorem provides the crucial step to proving a modified version of Littlewood's conjecture - a version representing the first significant advance on the Littlewood problem: for all but a set of dimension 0 of pairs α, β of real numbers, $\liminf_{n\to\infty} n\|n\alpha\| \ \|n\beta\| = 0$.

One of the seminal contributions of Lindenstrauss to this realm is his broadening of the notion of recurrence of a measure to a wide variety of situations, in particular, to situations where the measure is not invariant under a certain set of transformations. Quoting Lindenstrauss, "the only thing which is really needed is some form of recurrence which produces the complicated orbits which are the life and blood of ergodic theory."

This brings us to what is possibly the most exciting work of Elon Lindenstrauss; namely the solution of the Quantum Unique Ergodicity question in the arithmetic case. From the mathematical standpoint the issue is whether eigenfunctions of the Laplace operator on a negatively curved manifold tend to be more and more evenly spread over the space as the eigenvalue tends to negative infinity. In the special case of arithmetic hyperbolic surfaces, the so-called Hecke operators come into the picture and they act on the limiting measure arising from such a sequence of eigenfunctions. This action is recurrent and the tools developed by Lindenstrauss become applicable to this situation at hand, and lead elegantly to a solution of the problem.

Solving the so-called arithmetic quantum unique ergodicity conjecture of Rudnick and Sarnak is exciting if for no other reason than that the conjecture has been established provisionally, based on the generalized Riemann hypothesis. While this doesn't bring us closer to a solution of this famous question, this connection does testify to the depth of the mathematics involved.

I close my introductory remarks by mentioning one of the corollaries of Elon Lindenstrauss's handling of the arithmetic QUE conjecture; namely replacing reals by adèles and integers by rationals, we can speak of the adelic analogue of geodesic flow: namely, the action of the diagonal of $\mathrm{SL}_2(\mathbb{A})$ on $\mathrm{SL}_2(\mathbb{A})/\mathrm{SL}_2(\mathbb{Q})$. The striking statement is that the adelic geodesic flow is uniquely ergodic.

I think it is fair to say that there is both power and beauty in the mathematical work of Elon Lindenstrauss.