THE WORK OF STANISLAV SMIRNOV

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Stanislav (Stas for short) Smirnov is receiving a Fields medal for his ingenious and astonishing work on the existence and conformal invariance of scaling limits or continuum limits of lattice models in statistical physics.

Stas demonstrated his mathematical skills at an early age. According to Wikipedia he was born on Sept 3,1970 and was ranked first in the 1986 and 1987 International Mathematical Olympiads. He was an undergraduate at Saint Petersburg State University and obtained his Ph.D. at Caltech in 1996 with Nikolai Makarov as his thesis advisor. Stas has also worked on complex analysis and dynamical systems, and more physical models than percolation (e.g., Ising model).

Shall discuss his work on limits of lattice models. This work should make statistical physicists happy because it confirms rigorously what so far was accepted on merely heuristic grounds. The success of Stas in analyzing lattice models in statistical physics will undoubtedly be a stimulus for further work.

Wonderful result of Stas (together with Hugo Duminil-Copin, [11]). Announced only two months ago. Rigorous proof that the connective constant of the planar hexagonal lattice is $\sqrt{2+\sqrt{2}}$.

The *connective constant* μ of a lattice $\mathcal{L}$ is defined as $\lim_{n\to\infty}[c_n]^{1/n}$, where c_n is the number of self-avoiding paths on $\mathcal{L}$ of length n which start at a fixed vertex v. Usually easy to show by subadditivity (or better submultiplicativity; $c_{n+m} \le c_n c_m$) that this limit exists and is independent of the choice of v. However, the value of μ is unknown for most $\mathcal{L}$. This result of Stas is another major success in Statistical Physics.

1. Percolation

So far Stas best known for work in percolation, so discuss this now.

The first percolation problem appeared in Amer. Math Monthly, vol. 1 (1894), proposed by M.A.C.E. De Volson Wood ([10]). "An equal number of white and black balls of equal size are thrown into a rectangular box, what is the probability that there will be contiguous contact of white balls from one end of the box to he opposite end ? As a special example, suppose there are 30 balls in the length of the box, 10 in the width, and 5 (or 10) layers deep."

Only an incorrect solution. Hiatus till 1954.

Broadbent ([1]) asks Hammersley at a symposium on Monte-Carlo methods (in 1954): Think of the edges of $\mathbb{Z}^d$ as tubes through which fluid can flow with probability p, and are blocked with probability $1-p$. p is the same for all edges, and

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the edges are independent of each other. If fluid is pumped in at the origin, how far can it spread ? Can it reach infinity ?

Physicists interested in the model since it seems to be one of the simplest models which has a phase transition. In fact Broadbent and Hammersley ([1, 2]) proved that there exists a value p_c, *strictly* between 0 and 1, such that ∞ is reached with probability 0 when $p < p_c$, but can be reached with strictly positive probability for $p > p_c$. p_c is called the *critical probability.* The *percolation probability* $\theta(p)$ is defined as the probability that infinity is reached from the origin (or from any other fixed vertex).

Let E be a set of edges. Say that a point a is *connected* (in E) to a point b if there is an open path (in E) from a to b. Define the *open clusters* as maximal connected components of open edges in E. By translation invariance, Broadbent and Hammersley result shows that for $p < p_c$, with probability 1 all open clusters are finite; for $p > p_c$, (on $\mathbb{Z}^d$) with probability 1 there exists a unique infinite open cluster (see [3] for uniqueness). Can replace $\mathbb{Z}^d$ by another lattice. Also can have all edges open, but the vertices open with probability p and closed with probability $1 - p$. In obvious terminology, bond and site percolation. Site percolation is more general than bond percolation.

For Stas' brilliant result shall consider exclusively site percolation on the 2-dimensional triangular lattice.

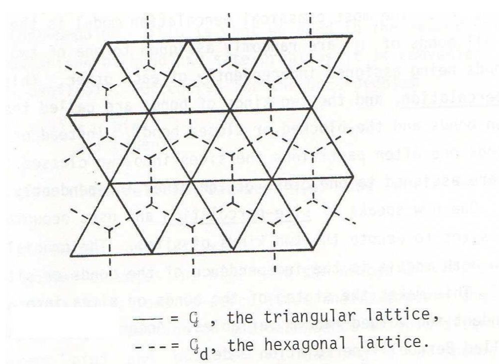

FIGURE 1

Would like to have a global (as opposed to microscopic) description of such systems. What is $\theta(p, \mathcal{L})$? Behavior of the "average cluster size" and some other functions. Have a fair understanding of the system for $p \neq p_c$ fixed. E.g., if $p < p_c$, then (with probability 1) there is a translation invariant system of finite clusters. The probability that the volume of the cluster of a fixed site exceeds n decreases exponentially in n (see [12], Theorem 6.75). If $p > p_c$, then there is exactly one infinite open cluster. If $\mathcal{C}$ denotes the open cluster of the origin, then for $p > p_c$ and some constants $0 < c_1(p) \leq c_2(p) < \infty$,

$$c_1 n^{(d-1)/d)} \leq -\log\left[P_p\{|\mathcal{C}|\right] = n\} \leq c_2 n^{(d-1)/d}.$$

For $d = 2$ we even know that

$$0 < -\lim_{n\to\infty} n^{-(d-1)/d} \log \big[P_p\{|\mathcal{C}|\big] = n\} < \infty,$$

i.e., for some $0 < c(p) < \infty$,

$$P_p\{|\mathcal{C}|\big] = n\} = \exp\big[-(c + o(1))n^{(d-1)/d}\big]$$

(see [12], Section 8.6).

Expect most interesting behavior for p equal or close to p_c.

We have here a system with a function $\theta(p, \mathcal{L})$, which has a phase transition, but, at least in dimension 2, is continuous. Physicists have been successful in analyzing such systems by use of so-called *scaling hypothesis*: for $p \neq p_c$ there is a single length scale $\xi(p)$, called the *correlation length*, such that for p close to p_c, at distance n the picture of the system looks like a single function of $n/\xi(p)$. More explicitly, it is assumed that many quantities behave like $T\big(n/\xi(p)\big)$ for some function T which is the same for a class of lattices $\mathcal{L}$. What happens when $p = p_c$? No special length scale singled out (other than the lattice spacing) ? The correlation length is believed to go to ∞ as $p \to p_c$. Therefore, shall think of looking at our system in a fixed piece of space, but letting the lattice spacing go to 0. Call this "taking the scaling limit" or "taking the continuum limit." Stas' great result: This limit exists and is conformally invariant for critical site percolation on the triangular lattice in the plane.

2. The scaling limit

What to expect when $p = p_c$? One hopes that at least the cluster distribution and the distribution of the interfaces (curves separating two adjacent clusters) converge in some sense in the scaling limit. Since there is no special scale, one expects scale invariance of the limit. If $\mathcal{L}$ has enough symmetry can also hope for rotational symmetry of scaling limit. In dimension two, scale and rotation invariance together should give invariance under holomorphic transformations. If one believes in scale invariance, then can expect power laws, i.e., that certain functions behave like a power of n or $|p - p_c|$ for n large or p close to p_c. E.g., if we set $R = R(p) =$ the radius of the open cluster of the origin, then scale invariance at $p = p_c$ would give

$$\frac{P_{p_c}\{R \geq xy\}}{P_{p_c}\{R \geq y\}} \to g(x) \tag{2.1}$$

for some function $g(x)$, as $y \to \infty$ and $x \geq 1$ fixed. This, in turn, would imply $g(xu) = g(x)g(u)$ and $g(x) = x^\lambda$ for some constant λ. Complete proof of (2.3) and evaluation of λ in [17] is much more intricate.

Necessarily $\lambda \leq 0$, since (2.1) ≤ 1 for $x \geq 1$. Now let $\varepsilon > 0$ and $(1+\varepsilon)^k \leq t \leq (1+\varepsilon)^{k+1}$. Then

$$P_{p_c}\{R \geq t\} \leq P_{p_c}\{R \geq (1+\varepsilon)^k\} = P_{p_c}\{R \geq 1\} \prod_{j=1}^{k} \frac{P_{p_c}\{R \geq (1+\varepsilon)^j\}}{P_{p_c}\{R \geq (1+\varepsilon)^{(j-1)}\}}. \tag{2.2}$$

Since

$$\frac{P_{p_c}\{R \geq (1+\varepsilon)^j\}}{P_{p_c}\{R \geq (1+\varepsilon)^{(j-1)}\}} \to g(1+\varepsilon) = (1+\varepsilon)^\lambda \text{ as } j \to \infty,$$

we obtain

$$P_{p_c}\{R \geq t\} \leq t^{\lambda + o(1)} \text{ as } t \to \infty.$$

By replacing k by $k+1$ and reversing the inequality in the lines following (2.2) we see that

$$P_{p_c}\{R \geq t\} = t^{\lambda + o(1)} \text{ as } t \to \infty \text{ or } \lim_{t\to\infty} \frac{\log P_{p_c}\{R \geq t\}}{\log t} = \lambda. \tag{2.3}$$

Of course we did not prove (2.1) here, nor did we obtain information about λ.

Different but related kind of power law

$$\frac{\log\big[\theta(p)\big]}{\log(p - p_c)} \to \beta \text{ as } p \downarrow p_c.$$

Exponents such as λ and β are called *critical exponents*. It is believed that all these exponents can be obtained as algebraic functions of only a small number of independent exponents. Physicists have indeed found (non-rigorously) that various quantities behave as powers. On basis of heuristics and simulations, exponents are believed to be *universal*: they depend basically on the dimension of the lattice only. Should exist and be the same for the bond and site version on $\mathbb{Z}^2$ and the bond and site version on the triangular lattice. For the planar lattices physicists predicted values for many of these exponents.

The pathbreaking work of Stas and Lawler, Schram, Werner has made proof of some power laws possible. E.g., processes related to site percolation on triangular lattice, loop erased random walk, uniform spanning tree. Nevertheless, no proof yet of universality for percolation. *Tacitly assume all further results for percolation on the triangular lattice.* As stated by Stas in his lecture at the last ICM ([22],p. 1421), "The point which is perhaps still less understood both from mathematics and physics points of view is why there exists a universal conformally equivalent scaling limit."

Many people believe that proving existence and conformal equivalence of scaling limit would be useful. This is still vague statement. Topology for scaling limit not clearly specified. It seems that M. Aizenman (see [15], bottom of p. 556) was the first to express the existence of the limit of crossing probabilities as a requirement for scaling limit.

A *crossing probability* of a Jordan domain $\mathcal{D}$ with boundary the Jordan curve $\partial\mathcal{D}$ is a probability of the form

$$P\{\exists \text{ a simple, occupied path in } \overline{\mathcal{D}} \text{ from the arc } [a,b] \text{ to the arc } [c,d]\},$$

where $\overline{\mathcal{D}} =$ closure of $\mathcal{D}$, and a, b, c, d are four points in counterclockwise order on $\partial\mathcal{D}$ and the interiors of the four arcs $[a,b], [b,c], [c,d]$ and $[d,a]$ are disjoint. We may replace "occupied path" by "vacant path" in this definition.

Seems reasonable to require that each crossing probability converges to some limit if our percolation configuration converges. We shall see soon that this is indeed the case in Stas' approach.

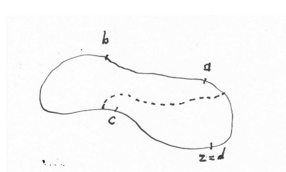

FIGURE 2

Now suppose $\mathcal{D}$ is a Jordan domain with boundary the Jordan curve J which contains the 4 points a, b, c, d (more generality is possible, but we don't discuss technicalities here). Consider limit of crossing probability from arc $[d, a]$ to arc $[b, c]$. Replace d by z and consider the function

$$H^\delta(a,b,c,z) := E\{Q^\delta(a,b,c,z)\}$$

as a function of $z \in \overline{\mathcal{D}}$, where

$$\begin{aligned} Q^\delta(z) = &\text{ there exists in } \overline{\mathcal{D}} \text{ a simple, occupied path from the} \\ &\text{arc } [b,c] \text{ to the arc } [c,a] \text{ which separates } z \text{ from the arc } [a,b]. \end{aligned} \tag{2.4}$$

Want to show that $\lim_{\delta \downarrow 0} H^\delta(a,b,c,z)$ exists and has value which agrees with Cardy's formula ([8]).

Stas realized that $H^\delta(\cdot)$ has to satisfy certain boundary conditions which we denote as (BC), without spelling them out.(A mixed Dirichlet-Neumann problem). It has a unique harmonic solution h, say. He had the great insight how to use uniqueness of *harmonic* functions which satisfy (BC) on $\overline{\mathcal{D}}$. By a sequence of ingenious tricks Stas manages to show that every subsequential limit of $H^\delta(a,b,c,z)$ (as $\delta \downarrow 0$) is harmonic in $z \in \overline{\mathcal{D}}$, and satisies (BC) and hence equals h. Since h is unique, the full limit $\lim_{\delta \downarrow 0} H^\delta(a,b,c,z)$ must exist and be equal to h. This limit is conformally invariant by construction.

By using Riemann mapping theorem can transfer calculation of h on $\mathcal{D}$ to calculation of h on an equilateral triangle. On such a triangle it is a linear function. Thus the crossing probabilities H^δ have limits, which can be computed explicitly. These limits agree with Cardy's formula ([8]).

This shows that individual crossing probabilities have a scaling limit which agree with Cardy's formula as desired. From here on there is still a lot of work to do to obtain the "full scaling limit". Stas [20, 21] outlined what needs to be done to get weak convergence of all long occupied and vacant paths. This was done by [5, 6, 7]. Later expositions in [4, 24]. Especially complicated is the picture of all loops. There are loops inside loops etc. As stated in the abstract of [5]: "These loops do not cross but do touch each other–indeed, any two loops are connected by a finite 'path' of touching loops."

3. Schramm-Loewner Evolutions (SLE).

A short time before Smirnov's paper, Schramm tried to find out how conformal invariance could be used (if shown to apply) to study also other models than percolation. Loewner tried with his evolutions to prove Bieberbach's conjecture. He represented a family of curves (one for each $z \in \mathbb{H}$) by means of a single function U_t. Here $\mathbb{H}$ is the open upper halfplane, U_t is a given function, and after a reparametrization, g_t is a solution of the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{3.1}$$

Let

$$T_z = \sup\{s: \text{ solution is well defined for } t \in [0,s) \text{ with } g_s(z) \in \mathbb{H}\}$$

and $H_t := \{z : T_z > t\}$. Then g_t is the unique conformal transformation from H_t onto $\mathbb{H}$ for which $g_t(z) - z \to 0$ as $z \to \infty$ (see [16], Theorem 4.6). The g_t arising in this way are called *(chordal) Loewner chains* and $\{U_t\}$ the *driving function*. See [16], Theorem 4.6.

Loewner chains and driving function $\{U_t\}$ were deterministic. Schramm [18] asked whether a random driving function could produce some of the known random curves as Loewner chain $\{g_t\}$. He showed in [18] that if the process $\{g_t\}$ has certain Markov properties, then one can obtain this process as Loewner chain only if the driving function is $\sqrt{\kappa} \times$ Brownian motion, for some $\kappa \geq 0$.

The processes which have such a driving function are called SLE's (originally this stood for "stochastic Loewner Evolution", but is now commonly read as Schramm-Loewner evolution).

When a chain is an SLE_κ, new computations become possible or much simplified. In particular, the existence and explicit values of most of the critical exponents have now been rigorously established (but see questions Q2 and Q4 below). Stas has made major contributions to these determinations in [17, 23]. In particular he provided essential steps for showing that a certain interface between occupied and vacant sites in percolation is an SLE_6 curve.

The SLE calculations confirm predictions of physicists, as well as a conjecture of Mandelbrot. Can now describe various planar random curves by means of SLE paths. Also intersection properties of Brownian paths. SLE_κ processes with different κ can have quite different behavior. A good survey of percolation and SLE is in [19], and [16] is a full length treatment of SLE.

4. Generalization and some open problems

I don't know of any lattice model in physics which has as much independence built in as percolation. It is therefore of great significance that Stas has a way to attack problems concerning the existence and conformal invariance of a scaling limit for some models with dependence between sites, and in particular for the two-dimensional Ising model. Stas and Dmitry Chelkak can apparently show that in the critical Ising model interfaces between spinclusters have a conformally invariant scaling limit which is described by SLE_3 curves. The oldest lattice model ? Enormous literature. I am largely ignorant of this literature and have not worked my way through Stas' papers on these models. Nevertheless am excited by the fact that Stas is seriously attacking such models.

To conclude, here are some problems on percolation. These also have appeared in other lists, (see in particular [19]),but you may like to be challenged again.

Q1 Prove the existence and find the value of critical exponents of percolation on other two-dimensional lattices than the triangular one and establish universality in two dimensions.

This seems to be quite beyond our reach at this time. Probably even more so is the same question in dimension > 2.

Q2 Prove a power law and find a critical exponent for the probability that there are j disjoint occupied paths from the disc $\{z : |z| \leq r\}$ to $\{z : |z| > R\}$. For $j = 1$ this is the one-arm problem of [17]. For $j \geq 2$, the problem is solved, at least for the triangular lattice, if some of the arms are occupied and some are vacant (see Theorem 4 in [23]), but it seems that there is not even a conjectured exponent for the case when all arms are to be occupied or all vacant.

More specific questions are

Q3 Is the percolation probability (right) continuous at p_c ? Equivalently, is there percolation at p_c ? This is only a problem for $d > 2$. The answer in $d = 2$ is that there is no percolation at p_c;

Q4 Establish the existence and find the value of a critical exponent for the expected number of clusters per site. This quantity is denoted by

$$\kappa(p) = \sum_{n=1}^{\infty} \frac{1}{n} P_p\{|C|\}$$

in [12], p. 23. The answer is still unknown, even for critical percolation on the two-dimensional triangular lattice. It is known that $\kappa(p)$ is twice differentiable on $[0, 1]$, but it is believed that the third derivative at p_c fails to exist; see [14], Chapter 9. This problem is mainly of historical interest, because there was an attempt to prove that p_c for bond percolation on $\mathbb{Z}^2$ equals $1/2$, by showing that $\kappa(p)$ has only one singularity in $(0, 1)$.

5. Conclusion

I have been amazed and greatly pleased by the progress which Stas Smirnov and coworkers have made in a decade. They have totally changed the fields of random planar curves and of two dimensional lattice models. Stas has shown that he has the talent and insight to produce surprising results, and his work has been a major stimulus for the explosion in the last 15 years or so of probabilistic results about random planar curves.

As some of the listed problems here show, there still are fundamental, and probably difficult, issues to be settled. I wish Stas a long and creative career, and that we all may enjoy his mathematics.

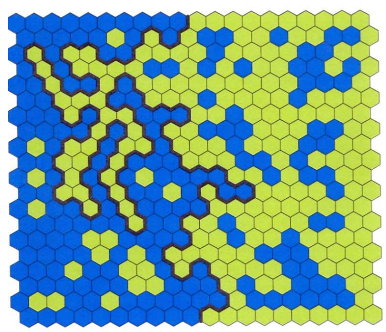

Figure 3

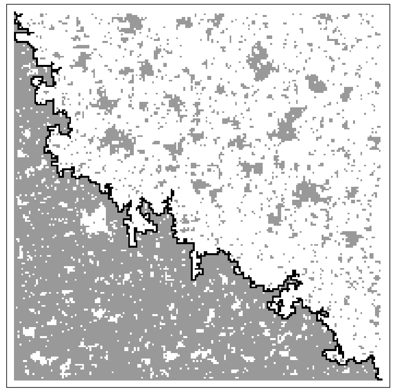

FIGURE 4

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