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Yves Meyer – Gauss Prize 2010

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Abstract. Yves Meyer has made numerous contributions to mathematics, several of which will be reviewed here, in particular in number theory, harmonic analysis and partial differential equations.
His work in harmonic analysis led him naturally to take an interest in wavelets, when they emerged in the early 1980s; his synthesis of the advanced theoretical results in singular integral operator theory, established by himself and others, and the requirements imposed by practical applications, led to enormous progress for wavelet theory and its applications. Wavelets and wavelet packets are now standard, extremely useful tools in many disciplines; their success is due in large measure to the vision, the insight and the enthusiasm of Yves Meyer.

Keywords. Harmonic analysis, wavelets, signal analysis, images, quasicrystals, Navier-Stokes

We start by reviewing the work by Yves Meyer chronologically, after which we comment on the many ways in which his work has had an impact outside mathematics.

1. Early work: Harmonic Analysis and Number Theory (1964-1973)

Although the Mathematics Genealogy Project lists Jean-Pierre Kahane as his Ph.D. advisor, Yves Meyer was essentially already an independent researcher when he wrote his PhD thesis, in which he solved a problem raised by Lennart Carleson about "strong Ditkin sets" [1]; this work constituted a precursor for later fundamental discoveries by Charles Fefferman and Elias Stein.

After his Ph.D., Meyer moved on to number theory, more precisely to Diophantine approximations. One of his early results was the construction of an increasing sequence of integers $(k_n)_{n\in\mathbb{N}}$ such that for any $t \in \mathbb{R}$, the sequence $(t\,k_n)_{n\in\mathbb{N}}$ is equidistributed *modulo* 1 if and only if t is transcendental [2]. This result preluded the characterization by Georges Rauzy of normal sets. Meyer also became interested in Pisot numbers and found a new approach to a theorem by Rafael Salem and Antoni Zygmund concerning sets of uniqueness of trigonometric expansions, proving in particular that certain types of Cantor sets have the property of spectral synthesis.

Insights gained while working on these early results then led to the first major contribution of Yves Meyer: the theory of *model sets*, which paved the road to the mathematical theory of *quasicrystals*.
A set $\Lambda \subset \mathbb{R}^n$ is a model set if there exist a finite set $F \subset \mathbb{R}^n$ and a constant $C \in \mathbb{R}_+$ such that (1) $\Lambda - \Lambda \subset \Lambda + F$, and (2) $\inf_{\lambda\in\Lambda} |x-\lambda| \leq C$ for all $x \in \mathbb{R}^n$. Meyer proved the following theorem: if Λ is a model set and if $\theta\,\Lambda \subset \Lambda$ then θ is a Pisot or a Salem number. The following converse is also true: for each Pisot or Salem number, there exists a model set Λ such that $\theta\,\Lambda \subset \Lambda$. This and many other properties of model sets are established in [3], relating them to the theory of mean-periodic functions developed by Jean Delsarte and Jean-Pierre Kahane. It was later realized that some non-periodic patterns observed in chemical alloys, now generally known as quasicrystals, could be identified with specific model sets. It is worth noting that these fundamental discoveries by Yves Meyer predated the first constructions of Penrose tilings came after these fundamental discoveries.

2. Singular integral operators: the Calderón program (1974-1984)

Alberto Calderón proposed to construct an improved pseudodifferential calculus, with minimal smoothness assumptions on the "symbol"; he introduced this generalization so as to

> *obtain stronger estimates and to prepare the ground for application to the theory of quasilinear and nonlinear differential operators.*

In particular, these new operators should include *singular integral operators*, of which the archetypical examples are the *Hilbert transform* H (in one dimension),

$$H(f)f(x) = \frac{1}{\pi} \lim_{\epsilon\to 0} \int_{y\in\mathbb{R},\, |x-y|>\epsilon} \frac{1}{x-y}\, f(y)\, dy \ ,$$

and the *Riesz transforms* R_i (in higher dimensions),

$$R_i(f)f(x) = \frac{1}{\pi} \lim_{\epsilon\to 0} \int_{y\in\mathbb{R}^n,\, |x-y|>\epsilon} \frac{x_i}{|x-y|^{n+1}}\, f(y)\, dy \quad .$$

More generally, a singular integral operator T in Caldéron-Zygmund theory is associated with a *kernel function* $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ in the following way:
for arbitrary smooth functions f, g, both compactly supported, and with disjoint supports,

$$\int_{\mathbb{R}^n} g(x)\, T(f)(x)\, dx \ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(x)\, K(x,y)\, f(y)\, dy\, dx \ ,$$

where K must satisfy some decay and cancellation conditions that nevertheless allow singular behavior of $K(x,y)$ as y approaches x. (More precisely, it is required that $|x-y|^n\,|K(x,y)|$ be uniformly bounded, and that, for some $\delta > 0$, and some $C \in \mathbb{R}_+$, $(|x-y|+|x'-y|)^n\ |K(x,y)-K(x',y)| \leq C\left(\frac{|x-x'|}{|x-y|+|x'-y|}\right)^{\delta}$, again uniformly in x, x' and y for $x \neq x'$, with a symmetric condition on $K(x,y)-K(x,y')$.)

The most famous examples of such operators are given by the Cauchy integral on a Lipschitz curve or the double layer potential on a Lipschitz surface. Together with Ronald Coifman and Alan McIntosh, Yves Meyer [4] obtained a breakthrough result in this framework, proving the boundedness of these Calderón-Zygmund operators for arbitrary Lipschitz curves or surfaces. This breakthrough opened the door for further fundamental results, such as the solution of the Dirichlet problem in arbitrary Lipschitz domains by the method of layer potentials [5], the celebrated $T(1)$ theorem of Guy David [6], proving boundedness of general Calderón-Zygmund operators under minimal conditions (generalized even further by David, Journé and Semmes [7]), and the solution of Katos conjecture about the square root of accretive differential operators [8].

One of the technical tools used repeatedly in the analysis of Calderón-Zygmund operators consists in integral formulas of the type

$$Q_s(f)(x) \;=\; s^n \int_{\mathbb{R}^n} f(x-y)\,q(sy)\,dy\ ,$$

where the function $q : \mathbb{R}^n \to \mathbb{R}$ is defined by $q(x) \;=\; 2^n\,\varphi(2x) - \varphi(x)$, for some well-localized and smooth "bump function" φ on $\mathbb{R}^n$, i.e. a smooth function with fast decay and integral 1, often picked radially symmetric for simplicity. One then easily checks that the following resolution of the identity holds, at least in the weak sense, and for reasonable f,

$$\int_0^\infty Q_s(Q_s(f))\frac{ds}{s} \;=\; C_\varphi\, f\ ,$$

where C_φ depends on the choise of φ, but not on f. The integral over the scaling parameter s can then be written as a sum over subsets of $\mathbb{R}_+$, carving up f into components at different scales.

3. Signal and image processing: the Wavelet revolution (1983-1993)

Wavelet theory finds its origin in the recurrent need to develop a localized version of Fourier analysis, inasmuch as is possible within the Heisenberg principle constraint.

Early attempts to obtain *time-frequency representations* for arbitrary (bounded) functions $f : \mathbb{R} \to \mathbb{C}$, via linear and bilinear transforms, were motivated at least in part by the desire to study the correspondence between classical and quantum mechanics: coherent state representations (already implicit in some of Schrödinger's work; introduced more explicitly by Gabor in 1945) which can be viewed as *short-time Fourier transforms* or *windowed Fourier transforms*,

$$S_w(f)(t,\omega) \;=\; \int_{\mathbb{R}} f(t+\tau)\, e^{i\omega\tau}\, w(\tau)\, d\tau$$

(where w is typically smooth and has compact support or fast decay), or the Wigner transform,

$$W(f,g)(t,\omega) \;=\; \int_{\mathbb{R}} f(t+\tau)\, \overline{g(t-\tau)}\, e^{2i\omega\tau}\, d\tau \quad ;$$

in this last case, $W(f,f)$ (in which $g=f$) is called the Wigner or Wigner-Ville distribution of f, first introduced in the 1930s. Figure 1 illustrates these time-frequency representations for one particular f.

In the windowed Fourier transform the extent of “time” or “frequency” localization is fixed in advance by the choice of the window function w. For instance, in Figure 1, the constant frequency component s_1 is clearly delineated in the windowed Fourier transform with wide window w_{wide}, and much less so when w_{narrow} is chosen; on the other hand, the temporal start of s_2 can be identified with greater accuracy in the windowed Fourier transform with w_{narrow} than with w_{wide}. One easily checks that, for a wide range of choices of f, g, including all f, $g \in \mathrm{L}^2(\mathbb{R})$,

$$\int_{\mathbb{R}\times\mathbb{R}} \overline{S_w(g)(t,\omega)}\, S_w(f)(t,\omega)\, dt\, d\omega \;=\; 2\pi \left[\int_{\mathbb{R}} |w(s)|^2\, ds\right] \int_{\mathbb{R}} \overline{g(\tau)}\, f(\tau)\, d\tau \;;$$

writing out explicitly the integrals in $S_w(g)$, one finds that this can be interpreted (in the weak sense) as

$$f(\tau) \;=\; (2\pi)^{-1} \int_{\mathbb{R}\times\mathbb{R}} S_w(f)(t,\omega)\, w(\tau-t)\, e^{-i\omega\tau}\, dt\, d\omega \;,$$

where we assume $\int_{\mathbb{R}} |w(s)|^2\, ds \;=\; 1$ for simplicity. With judicious choices of the window w and of parameters t_0, ω_0, there exist similar *decomposition formulas* using discrete sums rather than integrals, i.e.

$$f(\tau) \;=\; (2\pi)^{-1} \sum_{m,n\in\mathbb{Z}} S_w(f)(mt_0, n\omega_0)\, w(\tau - mt_0)\, e^{-in\omega_0\tau} \;.$$

The Gabor transform is exactly of this type, with a Gaussian window w. These integrals or sums can be viewed as ways to write f as a superposition of “atoms” $w_{[t,\omega]}(\tau) \;:=\; w(\tau-t)\, e^{-i\omega\tau}$ that are each well localized in time and frequency around their label $[t,\omega]$; note that each $w_{[t,\omega]}$ is obtained from the “generating” atom w by simple translation in time and in frequency. These decompositions suffer, however, from the shortcoming illustrated by Figure 1: the choice of the window fixes the trade-off between precision in time and frequency localization, which then remains the same throughout the time-frequency plane.

This shortcoming led Jean Morlet, a seismological engineer, to introduce a new integral transform based on time-scale atoms, generated by translates and *dilates* of an atom ψ, i.e. $\psi^{[a,t]}(\tau) := N_a\,\psi\left(\frac{\tau-t}{a}\right)$, where the normalization constant N_a can be adapted to the application at hand; often $N_a = a^{-1/2}$ is selected, ensuring a constant $L^2(\mathbb{R})$-norm for the $\psi^{[a,t]}$. Typically one picks ψ smooth, with fast decay; it is essential that it also satisfy $\int_{\mathbb{R}} \psi(t)\,dt = 0$. Analogously to the windowed or short-time Fourier transform $S_w(f)(t,\omega) = \int f(\tau)\,\overline{w_{t,\omega}(\tau)}\,d\tau$, one then defines the wavelet transform $T_\psi(f)$ by

$$T_\psi(f)(a,t) = \int_{\mathbb{R}} f(\tau)\,\overline{\psi^{[a,t]}(\tau))}\,d\tau\ .$$

The bottom right panel of Figure 1 illustrates that $T_\psi(f)$ does provide a time-frequency representation with high resolution in time for high frequency components, and high resolution in frequency for low frequency components.

The main theoretical properties of this transform were studied, in collaboration with Jean Morlet, by mathematical physicist Alex Grossmann. In particular, they showed that, in the same way as for the windowed Fourier transform, (and with the choice $N_a = a^{-1/2}$)

$$\int_{\mathbb{R}\times\mathbb{R}} \overline{T_\psi(g)(a,t)}\;T_\psi(f)(a,t)\,dt\,a^{-2}da = 2\pi\left[\int_{\mathbb{R}} |\xi|^{-1}\,|\widehat{\psi}(\xi)|^2\,d\xi\right]\int_{\mathbb{R}} \overline{g(\tau)}\,f(\tau)\,d\tau\ ,$$

where $\widehat{\psi}$ is the Fourier transform of ψ. Again, this can be interpreted as

$$f(\tau) = (2\pi)^{-1}\int_{\mathbb{R}\times\mathbb{R}} T_\psi(f)(a,t)\,\psi^{[a,t]}(\tau)\,dt\,a^{-2}da\ ,$$

with $\int_{\mathbb{R}} |\xi|^{-1}\,|\widehat{\psi}(\xi)|^2\,d\xi = 1$ for simplicity; judicious choices of ψ and of parameters t_0, a_0 lead to a similar discrete *decomposition formula*,

$$f = (2\pi)^{-1}\sum_{m,n\in\mathbb{Z}} T_\psi(f)(a_0^n,\,m\,a_0^n\,t_0)\,\psi^{[a_0^n,\,m\,a_0^n\,t_0]}\ .$$

These decomposition formulas for f turn out to be *exactly* the same as the formula at the end of the previous section (restricted to dimension 1): its integral over the scale s corresponds here to the integral over a, and the integral over t is just an explicit writing-out of the convolution inherent to the “outer” Q_s; the “inner” Q_s is subsumed in the wavelet transform $T_\psi(f)$.

Yves Meyer was the first to notice this similarity, and to realize that the wavelet transform proposed by Grossmann and Morlet was related to the very rich and powerful Calderón-Zygmund theory. He soon established that, in contrast to the windowed Fourier transform, the wavelet transform allows for discrete versions in which the $\psi^{[a_0^n,\,m\,a_0^n\,t_0]}$ constitute an orthonormal basis for $L^2(\mathbb{R})$. Several new families of bases, constructed by Meyer and by his student Pierre-Gilles Lemarié-Rieusset, as well as by the mathematical physicist Guy Battle, soon joined the two already existing constructions, by Alfred Haar and Jan-Olov Stromberg respectively, all featuring dyadically scaled functions of the type $\psi_{j,k}(t) = 2^{j/2}\,\psi(2^j tk)$, with j, k raging over $\mathbb{Z}$. In collaboration with Stéphane Mallat, Meyer constructed a general framework, *multiresolution analysis*, that not only provided the right setting to construct further wavelet bases, but also allowed the seamless integration of the new wavelet point of view with the existing Calderón-Zygmund theory. In particular, Meyer showed in his celebrated book [9] that these wavelet bases are unconditional bases for a host of classical function spaces; this is a key feature in many applications of wavelets, for instance in data compression and statistical estimation.

The work of Yves Meyer paved the way to the construction on orthonormal bases of compactly supported wavelets [10] and their subsequent biorthogonal generalization [11], corresponding to subband filtering algorithms with finite filters. The biorthogonal wavelet filters of [11] were selected as the filters of choice in the JPEG2000 image compression standard, recently adopted for the digital movies presently reaching movie theaters worldwide.

Wavelets have many more applications to science and technology, including denoising algorithms, adaptive numerical approximation of PDEs, medical and astronomical imaging, turbulence and genomic analysis.; a beautiful description of different perspectives can be found in the book [12] by Yves Meyer, Stéphane Jaffard and Robert Ryan. These applications are reflected by a large number of industrial patents, workshops, conference sessions and publications devoted to these applications.

In order to satisfy the requirements for an applications to astrophysics, Meyer, in collaboration with Ronald Coifman, extended the construction of wavelet bases to *wavelet packet bases*, which have since been used in numerous applications as well.

4. Navier-Stokes equations (1994-1999)

Yves Meyers interest in Navier-Stokes equation was inspired by a series of talks and papers by Marie Farge, as well as by a paper by Guy Battle and Paul Federbush., suggesting suggested that wavelet transforms might yield better results than pseudo-spectral algorithms for the numerical approximation of turbulent flow. This belief was grounded by the observation that turbulence involves a cascade of energy across a large range of scales and that wavelets provide a natural tool to identify the different scales and to analyze their interaction.

This led Yves Meyer to launch an research program on the Navier-Stokes equation, in collaboration with his students Marco Cannone, Fabrice Planchon and Pierre-Gilles Lemarié-Rieusset. It turned out that it is in fact more efficient to stick to Littlewood-Paley decompositions than to use wavelet expansions for the analysis of Navier-Stokes equations; using these decompositions they proved global existence of the solution in the space $C(\mathbb{R}_+, L^3(\mathbb{R}^3))$ when the initial condition u_0 is oscillating in the sense that it belongs to a Besov space of negative order; this was an improvement on the earlier Fujita-Kato theorem. A uniqueness result was later established by Pierre-Gilles Lemarié.

Another famous contribution of Yves Meyer to partial differential equations is an improved div-curl lemma, stating that if E and B are two square integrable vector fields such that $\nabla \cdot E$ and $\nabla \times B$ vanish, then $E \cdot B$ belongs to the Hardy space H^1. This remarkable result, first suggested by Pierre-Louis Lions, was proved by Yves Meyer and his collaborators in [13].

5. Recent work (2000-2008)

The results obtained by the group of Yves Meyer in nonlinear evolution equations led him to believe that there might be a functional norm governing the eventual blow-up of the solution to the Navier- Stokes equation. This endeavor ultimately led to dramatically improved Gagliardo-Nirenberg inequalities involving negative-regularity spaces, explaining why the solution of the Navier-Stokes equation does not blow up when the initial condition is oscillating. The study of these oscillatory patterns also led Yves Meyer back to the arena of image processing. A classical problem in image analysis is the separation of geometric features and texture. The algorithm proposed by Yves Meyer is based on a minimization procedure which involves the BV-norm to measure the geometric (or "cartoon") content and a negative smoothness norm to measure the oscillatory texture. This strongly improves on a celebrated algorithm proposed by Stanley Osher and Leonid Rudin. A comprehensive mathematical synthesis explaining the role of oscillation in both nonlinear partial differential equations and image processing was given by Yves Meyer in [14].

Most recently, Yves Meyer has been active in the field of compressed sensing. This very active field studies the extent to which one can exploit the inherent low-dimensional nature of an object or feature under study, when taking measurements in a high dimensional setting, when the identity of the "active" components is unknown. Based on abstract results from functional analysis and approximation theory from the 1960s, the fundamental estimates recently garnered an explosive amount of interest, after the work of Emmanuel Candès, Terrence Tao, David Donoho and many others who constructed concrete algorithms and illustrated their promise in applications.

A fundamental limitation in most approaches was that the best results were obtained with measurement matrices generated by probabilistic methods; typically deterministic constructions are less efficient. Yves Meyer gave the first deterministic construction of an optimal sensing system, based on the theory of model sets that he introduced at the start of his career, as well as a concrete algorithm for signal recovery from the measurements obtained by this system; in his approach the randomness is replaced by the pseudo-periodic structure generated by the model set.

6. Conclusions

The scientific life of Yves Meyer combines deep theoretical achievements in harmonic analysis, number theory, partial differential equations and operator theory, with a constant quest for a truly interdisciplinary exchange of ideas and the development of relevant and concrete applications.

This is illustrated most notably by his leading role in the development of wavelet theory, in which his research in harmonic analysis and operator theory led him naturally to the development of the computational multiscale methods that are at the heart of numerous applications of wavelets and wavelet packets in information science and technology.

His pioneering role is clear from the record. But to all his students and collaborators, Yves Meyer also stands out by other characteristics, maybe less tangible in the written record – his insatiable curiosity and drive to understand, his openness to other fields, his boundless enthusiasm and energy that inspired many young scientists, not all of them mathematicians, and the selfless generosity with which he untiringly promoted their work.

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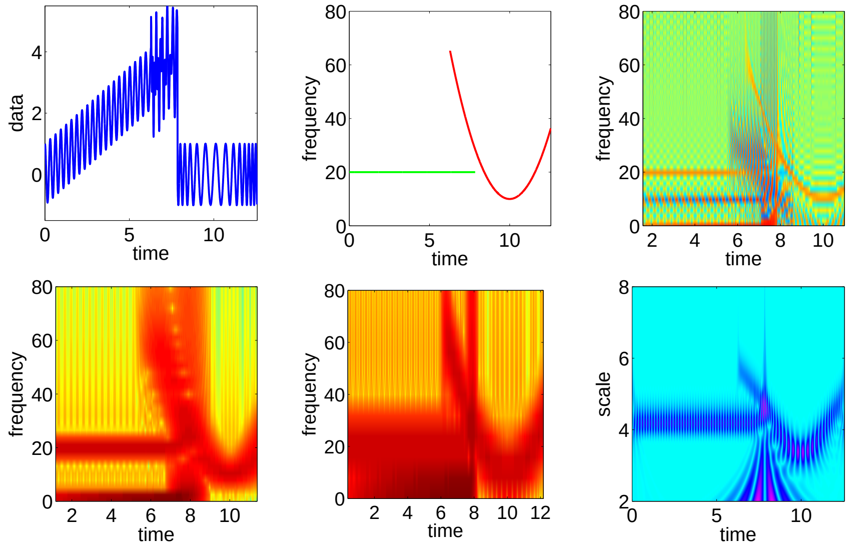

Figure 1. **Examples of time-frequency representations.** Top row left: the signal $f(t) = f_1(t) + f_2(t)$ defined by $f_1(t) = .5t + \cos(20t)$ for $0 \leq t \leq 5\pi/2$, and $f_2(t) = \cos\left(\frac{4}{3}\left[(t-10)^3 - (2\pi-10)^3\right] + 10(t-2\pi)\right)$ for $2\pi \leq t \leq 4\pi$; middle: the "instantaneous frequency" for its two components: for f_1, $\omega(t) = 20$ for $0 \leq (t-10)^2 \leq 5\pi/2$, and for f_2, $\omega(t) = 4t^2 + 10$ for $2\pi \leq t \leq 4\pi$; right: the Wigner-Ville distribution of f. Bottom row left: the (absolute value of a) a continuous windowed Fourier transform of $f(t)$, with a window $w_{\rm wide}$ with a wide (compact) support in t; middle: same, but now with a window $w_{\rm narrow}$ with a less wide (compact) support in t; right: the (absolute value of a) a continuous wavelet transform of $f(t)$, where ψ is the Morlet wavelet (essentially a modulated Gaussian).
The quadratic nature (in f) of the Wigner-Ville distribution causes "interference" terms in the time-frequency representation, avoided in linear time-frequency methods such as the windowed Fourier transforms.
The two windowed Fourier transforms show how the choice of the window influences the corresponding time-frequency representation; in the wavelet transform the fine scale at high frequencies, and the wider time support at lower frequencies make it possible to identify both the frequency of f_1 and the onset of f_2 with greater accuracy than in either of the windowed Fourier transform representations.