Cédric Villani

Citation: "For his proofs of nonlinear Landau damping and convergence to equilibrium for the Boltzmann equation."

Cédric Villani has provided a deep mathematical understanding of a variety of physical phenomena. At the center of much of his work is his profound mathematical interpretation of the physical concept of entropy, which he has applied to solve major problems inspired by physics. Furthermore, his results have fed back into mathematics, enriching both fields though the connection.

Villani began his mathematical career by re-examining one of the most shocking and controversial theories of 19th century physics. In 1872, Ludwig Boltzmann studied what happens when the stopper is removed on a gas-filled beaker and the gas spreads around the room. Boltzmann explained the process by calculating the probability that a molecule of gas would be in a particular spot with a particular velocity at any particular moment – before the atomic theory of matter was widely accepted. Even more shockingly, though, his equation created an arrow of time.

The issue was this: When molecules bounce off each other, their interactions are regulated by Newton's laws, which are all perfectly time-reversible; that is, in principle, we could stop time, send all the molecules back in the direction they'd come from, and they would zip right back into the beaker. But Boltzmann's equation is *not* time-reversible. The molecules almost always go from a state of greater order (e.g., enclosed in the beaker) to less order (e.g., spread around the room). Or, more technically, entropy increases.

Over the next decades, physicists reconciled themselves to entropy's emergence from time-reversible laws, and indeed, entropy became a key tool in physics, probability theory, and information theory. A key question remained unanswered, though: How quickly does entropy increase? Experiments and numerical simulations could provide rough estimates, but no deep understanding of the process existed.

Villani, together with his collaborators Giuseppe Toscani and Laurent Desvillettes, developed the mathematical underpinnings needed to get a rigorous answer, even when the gas starts from a highly ordered state that has a long way to go to reach its disordered, equilibrium state. His discovery had a completely unexpected implication: though entropy always increases, sometimes it does so faster and sometimes slower. Furthermore, his work revealed connections between entropy and apparently unrelated areas of mathematics, such as Korn's inequality from elasticity theory.

After this accomplishment, Villani brought his deep understanding of entropy to on another formerly controversial theory. In 1946, the Soviet physicist Lev Davidovich Landau made a mind-bending claim: In certain circumstances, a phenomenon can approach equilibrium without increasing entropy.

In a gas, the two phenomena always go together. Gas approaches equilibrium by spreading around a room, losing any order it initially had and increasing entropy as much as possible. But Landau argued plasma, a gas-like form of matter which contains so much energy that the electrons get ripped away from the atoms, was a different story. In plasma, the free-floating charged particles create an electrical field which in turn drives their motion. This means that unlike particles in a gas, which affect the motion only of other particles they happen to smash against, plasma particles influence the motion of far-away particles they never touch as well. That means that Boltzmann's equation for gases doesn't apply – and the Vlasov-Poisson equation that does *is* time-reversible, and hence does not involve an increase in entropy.

Nevertheless, plasma, like gas, spreads out and approaches an equilibrium state. It was believed that this happened only because of the collisions between atoms. But Landau argued that even if there were no collisions, the plasma would move toward equilibrium because of a decay in the electric field. He proved it, too – but only for a simplified linear approximation of the Vlasov-Poisson equation.

Despite a huge amount of study over the next six decades, little progress was made on understanding how this equilibrium state comes about or proving Landau's claim for the full Vlasov-Poisson equation. Last year, Villani, in collaboration with Clément Mouhot, finally came to a deep understanding of the process and proved Landau right.

A third major area of Villani's work initially seemed to have nothing to do with entropy – until Villani found deep connections and transformed the field. He became involved in optimal transport theory, which grew out of one of the most practical of questions: Suppose you have a bunch of mines and a bunch of factories, in different locations, with varying costs to move the ore from each particular mine to each particular factory. What is the cheapest way to transport the ore?

This problem was first studied by the French mathematician Gaspard Monge in 1781 and rediscovered by the Russian mathematician Leonid Kantorovich in 1938. Kantorovich's work on this problem blossomed into an entire field (linear programming), won him the Nobel Prize in economics in 1975, and spread into a remarkable array of areas, including meteorology, dynamical systems, fluid mechanics, irrigation networks, image reconstruction, cosmology, the placement of reflector antennas – and, in the last couple of decades, mathematics.

Villani and Felix Otto made one of the critical connections when they realized that gas diffusion could be understood in the framework of optimal transport. An initial configuration of gas particles can be seen as the mines, and a later configuration can be seen as the factories. (More precisely, it's the probability distribution of the particles in each case.) The further the gas particles have to move to go from one configuration to the other, the higher the cost.

One can then imagine each of these possible configurations as corresponding to a point in an abstract mountainous landscape. The distance between two points is defined as the optimal transport cost, and the height of each point is defined by the entropy (with low points having high entropy). This gives a beautiful way of understanding what happens as gas spreads out in a room: It is as though the gas rolls down the slopes of this abstract terrain, its configurations changing as specified by the points on the downward path.

Now suppose that a fan is blowing when you open the beaker of gas, so that the gas doesn't spread uniformly as it diffuses. Mathematically, this can be modeled by considering the space in which the gas is spreading to be distorted or curved. Villani and Otto realized that the curvature of the space where the gas spreads would translate into the topography of the abstract landscape. This connection allowed them to apply the rich mathematical understanding of curvature (in particular, Ricci curvature, which was critical in the recent solution of the Poincaré conjecture) to answer questions about optimal transport.

Furthermore, Villani and John Lott were able to take advantage of these links with optimal transport to further develop the theory of curvature. For example, mathematicians hadn't had a way of defining Ricci curvature at all in some situations, like at a sharp corner. Villani and Lott (and simultaneously, using complementary tools, Karl-Theodor Sturm) were able to use the connection with optimal transport to offer a definition and push the mathematical understanding of curvature to new, deeper levels. This depth of understanding and development of novel connections between different areas is typical of Villani's work.

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