Elon Lindenstrauss

Citation: “For his results on measure rigidity in ergodic theory, and their applications to number theory.”

Elon Lindenstrauss has developed extraordinarily powerful theoretical tools in ergodic theory, a field of mathematics initially developed to understand celestial mechanics. He then used them, together with his deep understanding of ergodic theory, to solve a series of striking problems in areas of mathematics that are seemingly far afield. His methods are expected to continue to yield rich insights throughout mathematics for decades to come.

Ergodic theory studies dynamical systems, which are simply mathematical rules that describe how a system changes over time. So, for example, a dynamical system might describe a billiard ball ricocheting around a frictionless, pocketless billiard table. The ball will travel in a straight line until it hits the side of the table, which it will bounce off of as if from a mirror. If the table is rectangular, this dynamical system is pretty simple and predictable, because a ball sent any direction will end up bouncing off each of the four walls at a consistent angle. But suppose, on the other hand, that the billiard table has rounded ends like a stadium. In that case, a ball from almost any starting position headed in almost any direction will shoot all over the entire stadium at endlessly varying angles. Systems with this kind of complicated behavior are called “ergodic.”

The way that mathematicians pin down this notion that the trajectories spread out all over the space is through the notion of “measure invariance.” A measure can be thought of as a more flexible way to compute area, and having an invariant measure essentially assures that if two regions of the space in some sense have equal areas, points will travel into them the same percentage of the time. By contrast, in the rectangular table (which of course is not ergodic), the center will get very little traffic in most directions.

In many dynamical systems, there is more than one invariant measure, that is, more than one way of computing area for which almost all the trajectories will go into equal areas equally often. In fact, there are often infinitely many invariant measures. What Lindenstrauss showed, however, is that in certain circumstances, there can be only a very few invariant measures. This turns out to be an extremely powerful tool, a kind of hammer that can break hard problems open.

Lindenstrauss then adroitly wielded his hammer to crack some hard problems indeed. One example of this is in an area called “Diophantine approximations,” which is about finding rational numbers that are usefully close to irrational ones. Pi, for example, can be approximated pretty well as 22/7. The rational number 179/57 is a bit closer, but because its denominator is so much larger, it’s not as convenient an approximation. In the early 19th century, the German mathematician Johan Dirichlet proposed one possible standard for judging the quality of an approximation: The imprecision of a rational approximation *p/q* should be less than ${}^{\square}\!/_{q}\,\square$ . He then went on to show, in a not very difficult proof, that there are infinitely many approximations to any irrational number that meet this standard.

(To put this in formula form, he showed that for any real number α, there are infinitely many integers *p* and *q* such that $\left|\alpha - p/q\right| < 1/q^2$.)

Eighty years ago, the British mathematician John Edensor Littlewood proposed an analogue to Dirichlet's statement to approximate two irrational numbers at once: It should be possible, he figured, to find approximations *p/q* to α and *r/q* to β so that the product of the imprecision of the two approximations would be as small as you please. (In formula form, the claim is that for any real numbers α and β and any tiny positive quantity ε you like, there will be approximations *p/q* to α and *r/q* to β so that $\left|\alpha - p/q\right| \times \left|\beta - r/q\right| < \varepsilon/q^3$.) He gave the problem to his graduate students, thinking it shouldn't be *that* much harder than Dirichlet's proof. But the Littlewood Conjecture turned out to be extraordinarily difficult, and until recently, no substantial progress had been made on it.

Then Lindenstrauss brought his ergodic theory tools to the problem, in joint work with Manfred Einsiedler and Anatole Katok. Ergodic theory might seem an odd choice for a problem that doesn't involve dynamical systems or time, but such unlikely pairings are sometimes the most powerful. Here's one way of reformulating Littlewood's problem to see a connection: First imagine a unit square, and glue the top edge to the bottom edge to make a cylinder. Now glue the right edge to the left edge and you'll get a shape called a torus that looks like a donut. You can roll up the entire coordinate plane to this same shape by gluing any point *(x, y)* to the point whose *x*-coordinate is the fractional part of *x* and whose *y*-coordinate is the fractional part of *y*. This torus is the space of our dynamical system. We can then define a transformation by taking any point (*x, y*) to another point (*x* + α, y + β). If α and β are irrational (or more precisely, not rationally related), this dynamical system will be ergodic. The Littlewood Conjecture then becomes the claim that you can make these trajectories suitably close to the origin by applying the transformation enough times. The number of times you apply the transformation becomes the denominator of the fractions approximating α and β.

Using a reformulation of the Littlewood Conjecture in terms of a more complex dynamical system, the team made a huge step of progress on the conjecture: They showed that if there are any pairs of numbers for which the conjecture is false, there are only a very few of them, a negligible portion of them all.

Another example of the power of Lindenstrauss's work is his proof of the first nontrivial case of the arithmetic quantum unique ergodicity conjecture. Ergodic systems come up frequently in physics, because as soon as you have three bodies interacting, for example, the system starts to behave in a somewhat ergodic fashion. But if those interactions happen at the quantum scale, you can't describe them with the ordinary tools of ergodic theory, because quantum theory doesn't allow for well-defined paths of points at well-defined positions; instead, you can only consider the *probability* that a point will exist in a particular position at a particular time. Analyzing such systems mathematically has proven extraordinarily difficult, and physicists have had to rely on numerical simulations alone, without a firm mathematical underpinning.

The quantum unique ergodicity conjecture says, roughly, that if you calculate area using the measure that's natural in classical dynamics, then as the energy of the system goes up, this probability distribution becomes more evenly distributed over the space. Furthermore, this measure is the *only* one for which that is true. Lindenstrauss was able to prove this in an arithmetic context for particular kinds of dynamical systems, creating one of the first major, rigorous advances in the theory of quantum chaos.

These are just two examples of Lindenstrauss's remarkable results. His methods, tools and insights are likely to yield many more results in the years to come.

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