

The 2010 Chern Medal Award

Louis Nirenberg of Courant Institute of Mathematical Sciences, New York University (NYU), has been selected to be the first recipient of the Chern Medal “*for his role in the formulation of the modern theory of non-linear elliptic partial differential equations and for mentoring numerous students and post-docs in this area*”.

Louis Nirenberg’s Work

Nirenberg is undoubtedly one of the most outstanding analysts of the 20th Century. His work has had major influence in the development of different areas of mathematics and their applications. In particular, he has been a leader in most of the developments in the theory of linear and non-linear partial differential equations (PDEs) and related areas of analysis, the basic mathematical tools of modern science. PDEs arise in physics and geometry when systems depend on several variables simultaneously and the most interesting ones are non-linear. The importance of PDEs is clear from the fact of the seven million dollar Millennium Problems of Clay Mathematics Institute (CMI), three are in or related to PDEs. Nirenberg’s work in PDEs is deep and fundamental. He developed intricate connections between analysis and differential geometry and applied them to the theory of fluid phenomena and other physical phenomena.

Nirenberg’s thesis concerned a fundamental issue in geometry: the solution to an embedding problem in differential geometry that Hermann Weyl had posed around 1916. *Given a Riemannian metric on a unit sphere, with positive Gauss curvature, can you embed this 2-sphere isometrically in 3-space as a convex surface?* The thesis itself was a window to Nirenberg’s main interests -- PDEs and especially elliptic PDEs. In giving a positive answer to Weyl’s poser, Nirenberg used the ideas due to Charles Morrey. He proved the problem by reducing it to a problem in non-linear PDE. This equation is also an elliptic PDE. (In elliptic PDEs, the coefficients satisfy an inequality and these have applications in almost all areas of mathematics, as well as numerous applications in physics. As with a general PDE, an elliptic PDE may have non-constant coefficients and be non-linear. The basic example of an elliptic PDE is Laplace’s equation.)

As mathematician J. Mawhin has pointed out in his recent tribute to Nirenberg, ellipticity is a keyword in Nirenberg’s mathematical work. More than one-third of Nirenberg’s articles contain the word ‘elliptic’ just in their title, and a much larger fraction deals with elliptic equations or systems. “There is hardly any aspect of those equations he has not considered,” he has said. Despite their variety, elliptic PDEs have a well-developed theory to which Nirenberg contributed in large measure. In the first year of his publication (1953) itself he published three further papers, two of which involved non-linear elliptic PDEs, to which he returned in later years.

Through the next 20 years, with several collaborators, he developed the theory of elliptic equations satisfying certain regularity criteria which he had formulated. While proofs for

the existence and uniqueness of weak solutions to elliptic problems were well known, Nirenberg addressed the much more difficult question of how regular (or well-behaved) the weak solution is. (A weak solution to a PDE is one for which the derivatives appearing in the equation may not all exist but which nevertheless is deemed to satisfy the equation in a certain mathematical sense.) Today, Nirenberg's *method of differences* for proving the interior and boundary regularity is part of a graduate student's learning in PDEs.

A high point in his research in this area is the extension of this, which was done in collaboration with Shmuel Agmon and Avnon Doulis on *a priori* estimates for general linear elliptic systems. This is one of the most widely quoted results in analysis. As Nirenberg has said, the objective was to obtain "general estimates for general systems under general boundary conditions." The citation for the Steele Prize for Lifetime Achievement, which he received from the American Mathematical Society (1994), in fact, said: "Nirenberg is a master of the art and science of obtaining and applying *a priori* estimates in all fields of analysis."

Essential questions about regularity for Navier-Stokes equations, whose solutions determine fluid motion, are still open and it is one of CMI's Millennium Problems. Among the best results today is the Nirenberg-Caffarelli-Kohn estimate of the measure of the set of singularities. Nirenberg feels that the problem will get solved soon but will require more input from harmonic analysis.

Building on earlier estimates of Alberto Calderón and Antoni Zygmund, he and Joseph Kohn introduced the concept of a pseudo-differential operator, a generalization of the partial differentiation operator, which is useful in addressing the question of regularity of solutions to elliptic boundary value problems. They were addressing a problem that involved singular integral operators (integral operators that are not mathematically defined at a point) and they needed facts about certain properties of singular integral operators, but these were not available in literature. So they developed what they needed and what resulted was the extremely useful notion of a pseudo-differential operator. This has helped generate an enormous amount of later work in the field. Another important work was the one with François Trèves on the solvability of general linear PDEs. Some other highlights are his research on the regularity of free boundary problems with David Kinderlehrer and Joel Spuck. Such problems have found a wide range of applications including flame propagation.

As mentioned earlier, Nirenberg, a pioneer in non-linear PDEs has returned his attention at various stages to fully non-linear equations to make striking breakthroughs. An example is the series of papers on the existence of smooth solutions of equations of Monge-Ampère type, which are a non-linear second order PDEs with special symmetries, with Luis Caffarelli and Spuck. Nirenberg's study of symmetric solutions of non-linear elliptic equations using moving plane methods with Basilis Gidas and Wei Ming Ni, and later with Henri Berestycki, is an ingenious application of maximum principle. Mawhin has called Nirenberg as "Paganini of the maximum principle". Thanks to Nirenberg's ideas and methods, this has developed into a beautiful theory and has led to applications in combustion theory.

The following famous quotes of Nirenberg are indicative of his approach to handling non-linearity -- the problem determines the method. In his invited lecture at the Stockholm International Congress of Mathematicians (ICM 1962), he said: "Most results for non-linear problems are still obtained via linear ones, despite the fact that the problems are non-linear and not because of it." In the same lecture he also said commenting on someone's result: "The non-linear character of the equation is used in an essential way; indeed, he obtains results because of the non-linearity and not despite it."

Nirenberg's wide range of interests included differential geometry and topology as well where he made significant contributions. In harmonic analysis a function of bounded mean oscillation (BMO) is a real-valued function whose mean oscillation is bounded (finite). Motivated by Fritz John's earlier work in elasticity, Nirenberg, in association with John, investigated for the first time the topology of the space of such functions and gave a precise definition for it. The space is also sometimes called John-Nirenberg space. The results were crucial for later work of Charles Fefferman on this function space. This function space has subsequently been used in many parts of analysis and in martingale theory. More recently, motivated by some non-linear problems in physics, Nirenberg returned to this field and, in collaboration with Haim Brezis, investigated the space of functions with vanishing mean oscillation (VMO). The VMO functions are continuous functions that vanish at infinity and their function space is actually a subspace of BMO. Nirenberg and Brezis extended the 'degree theory' in topology to mappings belonging to VMO, a result that took topologists by surprise.

A fundamental question in the study of complex manifolds is: *When is an almost complex structure given by a complex structure?* A manifold is a topological space which can be locally described in terms of simpler well understood spaces such as Euclidean. For a manifold to be a complex manifold (where operations with imaginary numbers can be defined) the existence of an almost complex structure is necessary, but not sufficient. (An almost complex manifold is a smooth manifold equipped with a structure that, roughly speaking, defines operations with imaginary numbers on each tangent space, the differentiable manifold attached to every point.) That is, every complex manifold is an almost complex manifold, but not vice-versa.

In 1957 Nirenberg with his student August Newlander proved a fundamental result that answered this question that had been open for many years. According to Nirenberg, André Weil and Shiing-Shen Chern had drawn his attention to the problem of proving integrability condition for almost complex structures. Its resolution paved the way for its use in the study of many aspects of complex manifolds, particularly deformation theory. Although the problem is linear, Newlander and Nirenberg's proof reduces the problem to a system of non-linear PDEs such that each equation involves derivatives with respect to only one complex variable. Nirenberg's tremendous insight into the properties of PDEs and his unique ability to connect PDEs, analysis and geometry runs through all of Nirenberg's work.

Inequalities have had a special attraction for Nirenberg and there are several important inequalities associated with his name and a very important result is the set of Gardiardo-Nirenberg inequalities. The AMS citation called it a "minor gem". Nirenberg's love for inequalities comes from his long association with Kurt Friedrichs at the Courant Institute

for Mathematical Sciences (CIMS) at New York University (NYU) where Nirenberg ended after doing his under graduation with major in physics and mathematics from McGill University in 1945. Though he wanted to physics, thanks to an excellent physics teacher at McGill, he first did mathematics on the advice of Richard Courant. Nirenberg has said that Friedrichs was a major influence on him and his views of mathematics very much formed his view.

“Friedrichs was a great lover of inequalities and that affected me very much. The point of view was that inequalities are more interesting than equalities, the identities,” Nirenberg said in an interview to AMS. Elsewhere he has said: “I love inequalities. So if somebody shows me a new inequality, I say: Oh! that is beautiful, let me think about it.” Though he had wanted to do his Ph. D. under him, he ended up doing under Jim Stoker, which he finished in 1949. But Friedrichs’ influence is clearly visible in Nirenberg’s choice of problems, which are often drawn from physics. He does not distinguish between “pure” and “applied” mathematics, an attitude that has resulted from his entire career being spent at CIMS “where there is just mathematics and people are interested in pure and applied problems”, a heritage of Courant and Friedrichs. Though Nirenberg has said that he is more of a problem solver and not one who develops theories, his approach to the problem have resulted in the formulation of theories in different areas of mathematics.

Nirenberg is recognized for his excellent lectures and lucid expository writing. He has published over 185 papers and has had 46 students and 245 descendents according to the Mathematics Genealogy Project. In each of the last 10 years, top 15 cited papers in mathematics include at least 2 of Nirenberg, according to MathSciNet. The fact that nearly ninety per cent of Nirenberg’s papers are written in collaboration shows how Nirenberg, for over six decades, has shared his knowledge and mathematical insight with mathematicians from all over the world. He is also known to be very generous and enthusiastically presents the results of other mathematicians in his lectures and survey papers.

Kohn has said this of him: “Nirenberg's career has been an inspiration; his numerous students, collaborators, and colleagues have learned a great deal from him. Aside from mathematics, Nirenberg has taught all of us the enjoyment of travel, movies, and gastronomy. An appreciation of Nirenberg also must include his ever-present sense of humour. His humour is irrepressible, so that on occasion it makes its way to the printed page.”

In his AMS interview Nirenberg said: “I wrote one paper with Philip Hartman that was elementary but enormous fun to do. That's the thing I try to get across to people who don't know anything about mathematics, what fun it is! One of the wonders of mathematics is you go somewhere in the world and you meet other mathematicians, and it's like one big family. This large family is a wonderful joy.”

R. Ramachandran

