The 2010 Gauss Prize

Yves Meyer, Professor Emeritus at École Normale Supérieure de Cachan, France, has been selected for the 2010 Gauss Prize *"for fundamental contributions to number theory, operator theory and harmonic analysis, and his pivotal role in the development of wavelets and multiresolution analysis"*.

Yves Meyer's Work

"Whenever you feel competent about a theory, just abandon it." This has been Meyer's principle in his over four decades of outstanding mathematical research work. He believes that only researchers that are newborn to a theme, can show imagination and have big contributions. In this sense, Meyer has had four distinct phases of research activity corresponding to his explorations in four disparate areas – quasicrystals, Calderón-Zygmund programme, wavelets and Navier-Stokes equation. The varied subjects that he has worked on are indicative of his broad interests. In each one of them Meyer has made fundamental contributions. His extensive work in each would suggest that he does not leave a field of research that he has entered until he is convinced that the subject has been brought to its logical end. It is as if Meyer appears on the scene, ties up various loose ends and gives a unifying picture of the existing disparate approaches, which lays the foundation for a proper theoretical framework that has the Meyer stamp on it and he leaves the scene.

The seeds for Meyer`s highly original approach in every branch of mathematics that he has ventured into were perhaps sown early in his career. He started his research career after having been a high school teacher for three years folowing his university education. He completed his Ph. D. in 1966 in just three years. ``I was my own supervisor when I wrote my Ph. D.,`` Meyer has said. This individualistic perspective to a problem has been his hallmark till this day.

In 1970 Meyer introduced some totally new ideas in harmonic analysis (a branch of mathematics that studies the representation of functions or signals as a superposition of some basic waves) that turned out to be not only useful in number theory but also in the theory of the so-called quasicrystals. There are certain algebraic numbers called the Pisot-Vijayaraghavan numbers and certain numbers known as Salem numbers. These have some remarkable properties that show up in harmonic analysis and Diophantine approximation (approximation of real numbers by rational numbers). For instance, the Golden Ratio is such a number. Yves Meyer studied these numbers and proved a remarkable result. Meyer's work in this area led to notions of Meyer and model sets which played an important role in the mathematical theory of quasicrystals.

Quasicrystals are space-filling structures that are ordered but lack translational --symmetry and are aperiodic order in general. Classical

theory of crystals allows only 2, 3, 4, and 6-fold rotational symmetries, but quasicrystals display 5-fold symmetry and symmetry of other orders. Just like crystals quasicrystals produce modified Bragg diffraction, but where crystals have a simple repeating structure, quasicrystals exhibit more complex structures like aperiodic tilings. Penrose tilings is an example of such an aperiodic structure that displays five-fold symmetry. Meyer studied certain sets in the *n*-dimensional Euclidean space (now known as a Meyer set) which are characterized by a certain finiteness property of its set of distances. Meyer's idea was that the study of such sets includes the study of possible structures of quasicrystals. This formal basis has now become an important tool in the study of aperiodic structures in general.

In 1975 Meyer collaborated with Ronald Coifman on what are called Calderón-Zygmund operators. The important results that they obtained gave rise to several other works by others, which have led to applications in areas such as complex analysis, partial differential equations, ergodic theory, number theory and geometric measure theory. This approach of Meyer and Coifman can be looked upon as the interplay between two opposing paradigms: the classical complex-analytic approach and the more modern Calderón-Zygmund approach, which relies primarily on real-variable techniques. Nowadays, it is the latter approach that dominates, even for problems that actually belong to the area of complex analysis.

The Calderón-Zygmund approach was the result of the search for new techniques because the complex-analytic methods broke down in higher dimensions. This was done by S. Mihlin, Calderón and A. Zygmund who investigated and resolved the problem for a wide class of operators, which we now refer to as singular integral operators or Calderón--Zygmund operators. These singular integral operators are much more flexible than the standard representation of an operator, according to Meyer. His collaborative work with Coifman on certain multilinear integral operators has proved to be of great importance to the subject. With Coifman and Alan MacIntosh he proved the boundedness and continuity of the Cauchy integral operator, which is the most famous example of a singular integral operator, on all Lipschitz curves. This had been a long-standing problem in analysis.

Meyer calls the research phase on wavelets, which have had a tremendous impact on signal and image processing, as having given him a second scientific life. A wavelet is a brief wave-like oscillation with amplitude that starts out at zero, increases and decreases back to zero, like what may be recorded by a seismograph or heart monitor. But in mathematics these are specially constructed to satisfy certain mathematical requirements and are used in representing data or other functions. As mathematical tools they are used to extract information from many kinds of data including audio signals and images. Sets of wavelets are generally required to analyze the data. Wavelets can be combined with portions of an unknown signal by the technique of convolution to extract information from the unknown signal.

Representation of functions as a superposition of waves is not new. It has existed since the early 1800s when Joseph Fourier discovered that he could represent other

functions by superposing sines and cosines. Sine and cosine functions have well defined frequencies but extend to infinity; that is, while they are localized in frequency, they are not localized in time. This means that although we might be able to determine all the frequencies in a given signal, we do not know when they are present. For this reason a Fourier expansion cannot represent properly transient signals or signals with abrupt changes. For decades scientists have looked for more appropriate functions than these simple sin and cosine functions to approximate choppy signals.

To overcome this problem several solutions have been developed in the past decades to represent a signal in the time and the frequency domain at the same time. The effort in this direction began in the 1930s with the Wigner transform, a construction by Eugene Wigner, the famous mathematician-physicist. Basically wavelets are building blocks of function spaces that are more localized than Fourier series and integrals. The idea behind the joint time-frequency representations is to cut the signal of interest into several parts and analyze each part separately with a resolution matched to its scale. In wavelet analysis, appropriate approximating functions that are contained in finite domains and thus become very suitable for analyzing data with sharp discontinuities.

The fundamental question that the wavelet approach tries to answer is how to cut the signal. The time-frequency domain representation itself has a limitation imposed by the Heisenberg uncertainty principle that both the time and frequency domains cannot be localized to arbitrary accuracy simultaneously. Therefore, unfolding a signal in the time-frequency plane is a difficult problem which can be compared with writing the score and listening to the music simultaneously. So groups in diverse fields of research developed techniques to cut up signals localized in time according to resolution scales of their interest. These techniques were the precursors of the wavelet approach.

The wavelet analysis technique begins with choosing a wavelet prototype function, called the *mother wavelet.* Time resolution analysis can be performed with a contracted, high-frequency version of the mother wavelet. Frequency resolution analysis can be performed with a dilated, low-frequency version of the same wavelet. The wavelet transform or wavelet analysis is the most recent solution to overcome the limitations of the Fourier transform. In wavelet analysis the use of a fully scalable modulated window solves the signal-cutting problem mentioned earlier. The window is shifted along the signal and for every position the spectrum (the transform) is calculated. Then this process is repeated several times with a slightly shorter (or longer) window for every new cycle. The result of this repetitive signal analysis is a collection of time-scale representations of the signal, each with different resolution; in short, multiscale resolution or multiresolution analysis. Simply put the large scale is the big-picture, while the small scale shows the details. It is like zooming in without loss of detail. That is, wavelet analysis sees both the forest and trees.

In geophysics and seismic exploration, one could find models to analyze waveforms propagating underground. Multi-scale decompositions of images were used in computer vision because the scale depended on the depth of a scene. In audio processing, filter banks of constant octave-bandwidth (dilated filters) applied to the analysis of sounds and speech and to handle the problem of Doppler shift multiscale analysis of radar signals were evolved. In physics multiscale decompositions were used in quantum physics by Kenneth G. Wilson for the representation of coherent states and also to analyze the fractal properties of turbulence. In neuro-physiology, dilation models had been introduced by the physicist G. Zweig to model the responses of simple cells in the visual cortex and in the auditory cochlea. Wavelet analysis would bring these disparate approaches together into a unifying framework. Meyer is widely acknowledged as one of the founders of wavelet theory.

In 1981, Jean Morlet, a geologist working on seismic signals had developed what are known as `Morlet wavelets`, which performed much better than the Fourier transforms. Actually Morlet and Alex Grossman, a physicist whom Morlet had approached to understand the mathematical basis of what he was doing, were the first to coin the term wavelet in 1984. Meyer heard about the work and was the first to realize the connection between Morlet`s wavelets and earlier mathematical constructs, such as the work of Littlewood and Paley used for the construction functional spaces and for the analysis of singular operators in the Calderón-Zygmund programme.

Meyer studied whether it was possible to construct an orthonormal basis with wavelets. (An orthonormal basis is like a coordinate system in the space of functions and, like the familiar coordinate axes, each base function is orthogonal to the other. With an orthonormal basis you can represent every function in the space in terms of the basis functions.) This led to his first fundamental result in the subject of wavelets in a Bourbaki seminar article which constructs a whole lot of orthonormal bases with Schwarz class functions (functions which have values only over a small region and decay rapidly outside). This article was a majotr breakthrough that enabled subsequent analysis by Meyer. ``Ìn this article,`` says Stéphane Mallat, ``the construction of Meyer had isolated the key structures in which I could recognize similarities with the tools used in computer vision for multiscale image analysis and in signal processing for filter banks.``

A Mallat-Meyer collaboration resulted in the construction of mathematical multiresolution analysis, and a characterization of wavelet orthonormal bases with conjugate mirror filters that implement a first wavelet transform algorithm that performed faster than the Fast Fourier Transform (FFT) algorithm. Thanks to the Meyer-Mallat result, wavelets became much easier to use. One could now do a wavelet analysis without knowing the formula for the mother wavelet. The process was reduced to simple operations of averaging groups of pixels together and taking differences, over and over. The language of wavelets also became more comfortable to electrical engineers.

In the eighties digital revolution was all around and efficient algorithms were critically needed in signal and image processing. The JPEG standard for image compression was developed at that time. In 1987, Ingrid Daubechies, a student of Grossman, while visiting the Courant Institute at New York University and later during her appointment at AT&T Bell Labs, discovered a particular class of compactly supported conjugate mirror filters, which were not only orthogonal (like Meyer`s) but were stable and which could be implemented using simple digital filtering ideas. The new wavelets were simple to programme and they were smooth functions unlike some of the earlier jumpy functions. Signal processors now had a dream tool: way to break up digital data into contributions of various scales.

Combining Daubechies` and Mallat`s ideas, one could do a simple orthogonal transform that could be rapidly computed in modern digital computers. Daubechies wavelets turn the theory into a practical tool that can be easily programmed and used by a scientist with a minimum of mathematical training. Meyer`s first Bourbaki paper actually laid the foundations for a proper mathematical framework for wavelets. That marked the beginning of modern wavelet theory. In recent years wavelets have begun to provide an interesting alternative to Fourier transform methods.

Interestingly, Meyer`s the first reaction to the work of Grossman and Morlet was ``So what! We harmonic analysts knew all this a long time ago!`` But he looked at the work again and realized that Grossman and Morlet had done something different and interesting. He built on the difference to eventually formulate his basis construction. ``Meyer`s construction of the orthonormal bases and his subsequent results in the area were the key discovery that opened the door to all further mathematical developments and applications. Meyer was at the core of the catalysis that brought together mathematicians, scientists and engineers that built up the theory and resulting algorithms,`` says Mallat.

Since the work of Daubechies and Mallat applications that have been explored include multiresolution signal processing, image and data compression, telecommunications, fingerprint analysis, statistics, numerical analysis and speech processing. The fast and stable algorithm of Daubechies was improved subsequently in a joint work between Daubechies and Albert Cohen, Meyer`s student, which is now being used in the new standard JPEG2000 for image compression and is now part of the standard toolkit for signal and image processing. Techniques for restoring satellite images have also been developed based on wavelet analysis.

More recently, he has found a surprising connection between his early work on the model sets used to construct quasicrystals -- the 'Meyer Sets' -- and 'compressed sensing', a technique used for acquiring and reconstructing a signal utilizing the prior knowledge that it is sparse or compressible. Based on this he has developed a new algorithm for image processing. A version of such an algorithm has been installed in the space mission Herschel of the European Space Agency (ESA), which is aimed at providing images of the oldest and coldest stars in the universe.

`To my knowledge,`` says Wolfgang Dahmen, ``Meyer has never worked directly on a concrete application problem.`` Thus Meyer`s mathematics provide good examples of how the investigations of fundamental mathematical questions often yield surprising results that benefit humanity.

R. Ramachandran