Stability and recursive solutions in Hamiltonian PDEs

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Non linear PDE's

Non linear PDE's are ubiquitous in Mathematics and Physics. Some of the most famous examples as the Euler or Navier Stokes equations come from Hydrodynamics, as well as the Korteweg de Vries, Non Linear Schrödinger, Camassa Holm etc.

Why Non–linearity?

Non linearity describes mathematically self interactions of waves, fields etc..

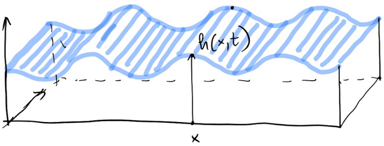
Non linear PDE's

Here are some well known examples in $1d$:

$$\text{(NLS)} \qquad u_t = -\mathrm{i}(u_{xx} + |u|^2 u)\,,$$

$$\text{(KdV)} \qquad u_t = u_{xxx} + uu_x\,,$$

$$\text{(CH)} \qquad u_t - u_{xxt} - u_x + u_{xxx} + uu_{xxx} + 3uu_x - 2u_x u_{xx} = 0$$



These are in fact all integrable PDEs which are widely studied as models in hydrodynamics.
The phenomena that I shall discuss do not rely on an integrable structure! (you could add a small perturbation!)

One is interested also in modeling waves in higher dimensional domains

$$\begin{cases} -i\partial_t u + \Delta u = |u|^{2p} u, \\ u = u(t,x)\,, \quad x \in D \end{cases} \qquad (NLS)$$

D can be a domain in $\mathbb{R}^n$, or a Riemannian manifold..

I will concentrate on PDEs defined on compact manifolds without boundary , mainly tori

$$\mathbb{T}^n := \mathbb{R}^n/(2\pi\mathbb{Z})^n\,, \quad u(\cdot, x) \text{ is } 2\pi \text{ periodic in each } x_i$$

When treating non–linear phenomena one should expect a very complex behavior with instability, chaos etc. (as well known to meteorologists!).

Many interesting models have some symmetry such as:

Hamiltonian or Reversible structure

and this leads to a coexistence of regular and chaotic orbits

I will concentrate on regular behavior.

consider PDEs on tori $\mathbb{T}^n$ close to an elliptic fixed point and study existence and stability of invariant tori taking a dynamical system point of view.

Dynamical systems in ∞ dimension.

we consider a non-linear dynamical system:

$$\dot{u} = F(u)\,, \qquad u \in V$$

where

- V is a vector space (in our case a scale of Hilbert spaces)
- F is a non-linear functional from V in itself.

PDE examples.

All the PDEs in my examples "fit this setting", for example

$$-i\partial_t u + \Delta u = |u|^2 u, \quad , \quad x \in \mathbb{T}^n$$

recalling

$$V \equiv \cup_{p\in\mathbb{R}} H_p := \{u(x) = \sum_{j\in\mathbb{Z}^n} u_j e^{\mathrm{i} j\cdot x}\,, \quad \sum_j \langle j\rangle^{2p} |u_j|^2 < \infty\}$$

$(\langle j\rangle := \max(1,|j|))$
we get

$$\dot u_j = \mathrm{i}|j|^2 u_j + \mathrm{i} \sum_{j_1-j_2+j_3=j} u_{j_1}\bar u_{j_2} u_{j_3}$$

If we consider $u_t - u_{xxt} - u_x + u_{xxx} + uu_{xxx} + 3uu_x - 2u_x u_{xx} = 0$
we get

$$\dot u_j = \mathrm{i}\lambda_j u_j + P_j\,, \quad \lambda_j \in \mathbb{R}$$

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we get

$$\dot u_j = \mathrm{i}\lambda_j u_j + P_j , \quad \lambda_j \in \mathbb{R}$$

Dynamical systems in ∞ dimension.

I shall study my system close to an elliptic fixed point that is

$$\dot{u} = F(u) = Lu + P(u)$$

where L is a (typically unbounded) linear operator with pure point purely immaginary spectrum
while P is a non-linear term (which has a zero of degree at least two at $u = 0$).
Thus one can reduce to a system of the form

$$\dot{u}_j = \mathrm{i}\lambda_j u_j + P_j(u)\,, \quad u = \{u_j\}_{j\in I}$$

where $\lambda_j \in \mathbb{R}$. This is a chain of harmonic oscillators coupled by a non-linearity.

Invariant subspaces for $\dot{u} = F(u)$

We look for a submanifold $\mathcal{M}$ of the phase space which is invariant for the system and on which the dynamics simplifies.

Recall that $\mathcal{M}$ is an invariant manifold for $\dot{u} = F(u)$, if for any initial datum $u_0 \in \mathcal{M}$ the solution $\Phi_t(u_0)$ stays in $\mathcal{M}$ for all times.

$\mathcal{M}$ is d dimensional invariant torus if there exists a diffeomorphism

$$U : \mathbb{T}^d \to \mathcal{M} \subset \text{the phase space}\,, \quad \theta \mapsto U(\theta)$$

where $\mathcal{M} = U(\mathbb{T}^d)$ is invariant...

Invariant tori

An invariant torus

$$U : \mathbb{T}^d \to \mathcal{M} \subset \text{the phase space}\,, \quad \theta \mapsto U(\theta)$$

is the support of a quasi-periodic solution of frequency $\omega \in \mathbb{R}^d$ if:

- The ω_i are rationally independent
- for all $\theta \in \mathbb{T}^d$, $u(t) = U(\theta + \omega t)$ solves $\dot{u} = F(u)$.

Infinite tori

One may also look for infinite dimensional invariant tori.

$$U : \mathbb{T}^{\mathbb{Z}} \quad \to \quad \mathcal{M} \subset \text{the phase space}$$

on which the dynamics is conjugated to the linear one $\varphi \to \varphi + \omega t$ with $\omega \in \mathbb{R}^{\mathbb{Z}}$

We are using $\mathbb{Z}$ as an index set because it is convenient in our examples!

Of course for infinite tori one needs to specify the topology on $\mathbb{T}^{\mathbb{Z}}$. We endow $\mathbb{T}^{\mathbb{Z}}$ with a Banach manifold structure by setting

$$\text{dist}(\theta, \varphi) = \sup_{j \in \mathbb{Z}} |\theta_j - \varphi_j|_{2\pi}$$

The topology of the phase space plays an important role!

Perturbation Theory

We shall mostly work in a perturbative setting...

- given an approximately invariant torus $\theta \to u_0(\theta)$, are we able to prove the existence of an invariant one close by?
- assume that $F = N_0 + P$ where N_0 admits invariant tori and P is small...do the invariant tori persist?

Once we have found an invariant torus we would like to understand the evolution of initial data close to the torus.

Some interesting questions

- (Stability) Prove that all initial data which start close to the torus stay close for finite but long times
- (Instability) Suppose that we have two invariant tori, construct solutions which drift from one torus to the other ...estimate the time over which such drift occurs

Small solutions

In the model

$$\dot{u}_j = \mathrm{i}\lambda_j u_j + P_j(u)\,, \quad u = \{u_j\}_{j\in I}$$

the simplest invariant object is $u = 0$.
There is a vast literature on stability/instability of elliptic fixed points!
Stability: Bourgain, Bambusi, Delort, Faou,Grebert, Szeftel, Yuan-Zhang, Cong-Mi-Wang, many more ....
Instability: Kuksin, Colliander-Keel-Staffilani-Takaoka-Tao, many more...

On PDEs without spatial confinement again a vast literature on stability/instability of zero. very different phenomena

Linear theory

If take u small, $P(u)$ is perturbative w.r.t. the linear terms...

$$\dot{u}_j = \mathrm{i}\lambda_j u_j + P_j(u)$$

The unperturbed system is a chain of uncoupled harmonic oscillators with frequencies λ_j.

$$\dot{u}_j = \mathrm{i}\lambda_j u_j \Rightarrow u_j(t) = e^{\mathrm{i}\lambda_j t} u_j(0)$$

All the solutions are on invariant tori with dimension depending on the support of the solution

$$\mathcal{S} := \{j \in \mathbb{Z}^n : \quad u_j(0) \neq 0\}$$

and on whether the λ_i are rationally independent.

$$u(t,x) = \sum_j u_j(0) e^{\mathrm{i}\lambda_j t + \mathrm{i}j\cdot x} = \sum_{j\in\mathcal{S}} \sqrt{\xi_j} e^{\mathrm{i}\lambda_j t + \mathrm{i}j\cdot x}$$

which solutions of the linear system survive the onset of the non-linearity?

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KAM theorems in finite dim. $\dot{u}_j = \mathrm{i}\lambda_j u_j + P_j$

In finite-dimensional nearly integrable Hamiltonian systems:

- Under some non-degeneracy assumptions (on λ or on P), the majority of initial data give rise to quasi-periodic solutions that densely fill some invariant torus and are, therefore, perpetually stable.
- The KAM theorems predict persistence of most but not all quasi–periodic orbits. Obstacles arise from *small divisors* and *resonances* which should be avoided by modulating the initial data. For example for maximal KAM tori a good condition is that the frequency should be Diophantine.
- In the complementary set to the quasi–periodic orbits one may see chaotic behavior .

KAM in infinite dimension

- In ∞-dim the scene is so far rather obscure: in an integrable PDE, almost-periodic solutions are typical and lie on maximal infinite dimensional invariant tori. what is their fate after perturbation? Is it still true that the majority of initial data produces perpetually stable solutions?
- There is a wide literature for existence of quasi-periodic solutions mostly for semilinear PDEs such solutions are NOT typical and correspond to lower dimensional tori
- there are very few results on infinite dimensional tori mostly for not very natural models.
- There are examples of PDEs which exhibit diffusive orbits.

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KAM for PDEs

The first results were on model Hamiltonian PDEs such as the semilinear NLS with Dirichlet boundary conditions

$$-\mathrm{i}u_t + u_{xx} + |u|^2 u + g(x,u) = 0, \quad u(t,0) = u(t,\pi) = 0$$

- *(Semilinear PDEs with Dirichlet b.c.* : Kuksin, Wayne, Pöschel, Kuksin-Pöschel, Periodic b.c. Chierchia-You , Craig-Wayne '93 (periodic solutions), Bourgain '94 (quasi periodic solutions),.
- Higher dimensional manifolds:
  Tori: Bourgain '98,'05, Wang '10-'15, Berti-Bolle '10- '15, Geng-You, Eliasson-Kuksin '10, Geng-You-Xu '10, Procesi-P. '11-'15,
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Some more literature: unbounded non linearities

- **semi-linear Pde's**
  Kuksin '98, Kappeler-Pöeschel '03 KdV, Liu-Yuan '10, Zhang-Gao-Yuan '11 Hamiltonian and Reversible DNLS Berti, Biasco, P. , Hamiltonian and Reversible DNLW
- **Quasi-linear or fully non-linear Pde's**
  periodic solutions: Ioss-Plotnikov-Toland '01- '10, Alazard,Baldi capillary water waves
  quasi-periodic solutions: Baldi, Berti, Montalto '11 Airy,Baldi, Berti, Haus, Montalto, Feola-Giuliani Berti-Kappeler-Montalto
  Higher Dimension: Corsi-Montalto, Baldi-Montalto, Feola-Grebert

Main difficulties. $\dot{u}_j = \mathrm{i}\lambda_j u_j + \varepsilon P_j(u)$

Most proofs use some quadratic iterative method to construct solitions.

I will focus on three types of difficulties:

- Resonances. In most natural PDEs the λ_j are rationally dependent and thus non generic
  The idea is to find find a familiy of appropriate approximately invariant tori. typically of the form

  $$\theta \to u_0(\theta) = \sum_{h=1}^{d} \sqrt{\xi_i} e^{\mathrm{i}\theta_h} e^{\mathrm{i}\mathtt{j}_h x}$$

  Where the $\{\mathtt{j}_1, \dots, \mathtt{j}_d\} \subset \mathbb{Z}^n$ are Fourier modes and ξ_i are parameters which are modulated to deal with small divisors.
  - $\{\mathtt{j}_1, \dots, \mathtt{j}_d\} \subset \mathbb{Z}^n$ generic (most choices in a big enough ball)
  - ξ_i are in some complicated set of positive measure in $\mathbb{R}^d$
- Multiplicity.
- Derivatives in the non-linearity The non-linear terms can

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I will focus on three types of difficulties:

- Resonances.
- Multiplicity. In most PDEs on compact manifolds the eigenvalues λ_j are multiple...
- Derivatives in the non-linearity The non-linear terms can contain as many space derivatives as the linear ones.

Main difficulties. $\dot{u}_j = \mathrm{i}\lambda_j u_j + \varepsilon P_j(u)$

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A result on the reversible autonomous NLS

Consider a reversible NLS equation

$$-\mathrm{i}u_t + u_{xx} + f(u, u_x, u_{xx}) = 0 \tag{1}$$

where

$$f(u, u_x, u_{xx}) = \mathtt{a}_1|u|^2u + \mathtt{a}_2|u|^2u_{xx} + \mathtt{a}_3|u_x|^2u + \mathtt{a}_4|u_x|^2u_{xx} + \mathtt{a}_5|u_{xx}|^2u + \mathtt{a}_6|u_{xx}|^2u_{xx}$$

with $\mathtt{a}_i \in \mathbb{R}$ for $i = 1, \dots, 6$. Suppose that

$$(\mathtt{a}_1, \mathtt{a}_2, \mathtt{a}_3, \mathtt{a}_4, \mathtt{a}_5, \mathtt{a}_6) \neq (0, a, a, b, b, 0)$$

Theorem (Corsi,Feola,P. 17)

For any generic choice of tangential sites $\mathtt{j}_1, \dots, \mathtt{j}_d \in \mathbb{Z}$ and for all $\varepsilon \in (0, \varepsilon_0)$, there exists a positive measure set

$$\mathcal{C}_\varepsilon \subset \varepsilon\,[1,2]^d\,, \quad |\mathcal{C}_\varepsilon|/\varepsilon^d \to 1 \quad as \quad \varepsilon \to 0, \tag{2}$$

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such that for all $\xi \in \mathcal{C}_\varepsilon$ *the NLS has a quasi- periodic solution*

$$\sum_{i=1}^{d} \sqrt{\xi_i} e^{\mathrm{i}\omega_i t + \mathrm{i}\mathtt{j}_i x} + O(\xi)\,, \quad \omega_i(\xi) = \mathtt{j}_i^2 + O(\xi)$$

The quasi-periodic solutions are analytic and linearly stable.

We also have a higher dimensional analogue.

$$-\mathrm{i}u_t + \Delta u - F(|u|^2)u = 0$$

Theorem (Procesi-M.P. 15)

For any generic choice of tangential sites $\mathtt{j}_1, \dots, \mathtt{j}_d \in \mathbb{Z}^n$ and for all $\varepsilon \in (0, \varepsilon_0)$, there exists a positive measure set

$$\mathcal{C}_\varepsilon \subset \varepsilon\,[1,2]^d\,, \quad |\mathcal{C}_\varepsilon|/\varepsilon^d \to \frac{1}{2} \quad \textit{as} \quad \varepsilon \to 0, \tag{3}$$

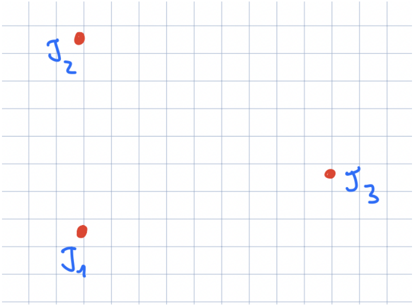
such that for all $\xi \in \mathcal{C}_\varepsilon$ the NLS has a quasi- periodic solution

$$\sum_{i=1}^{d} \sqrt{\xi_i} e^{\mathrm{i}t\omega_i(\xi)+\mathrm{i}\mathtt{j}_i\cdot x} + O(\xi)\,, \quad \omega_i = \omega_i(\xi) = |\mathtt{j}_i|^2 + O(\xi)$$

The quasi-periodic solutions are analytic and linearly stable.

Generic tangential sites.

In these results, the sites $\mathtt{j}_1, \ldots, \mathtt{j}_d \in \mathbb{Z}^n$ are generic in the sense that they do not satisfy a certain polynomial.
If $n = 2$ [GengYouXu] give arithmetic conditions:



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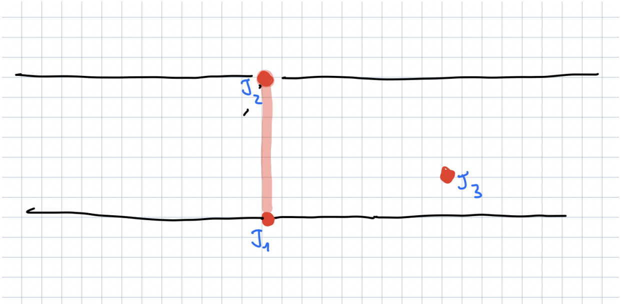

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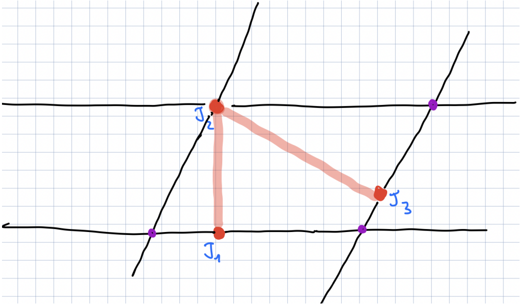

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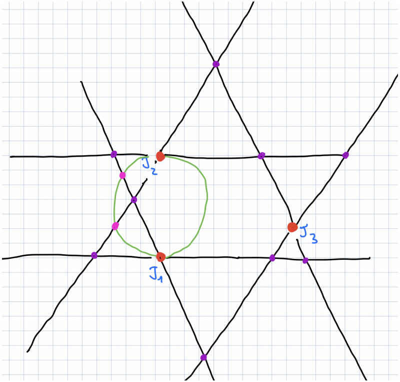
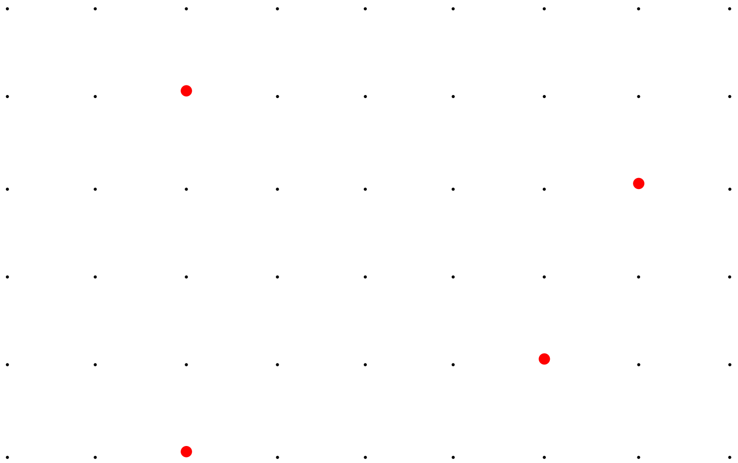
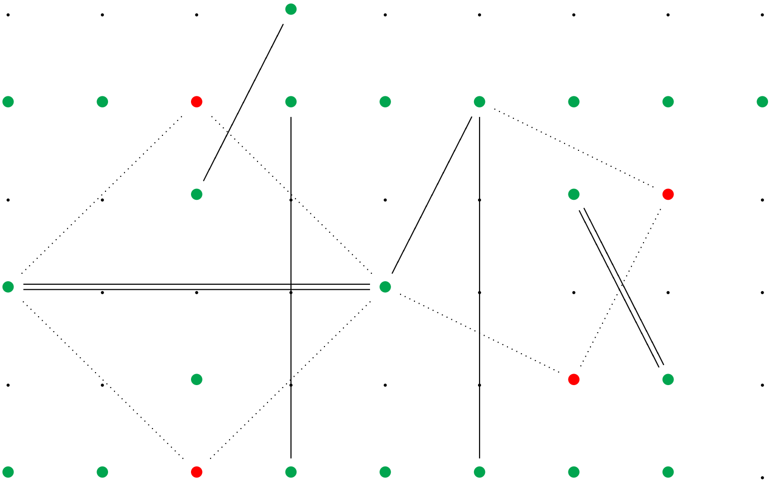


EXAMPLE: S is given by 4 points marked ●



EXAMPLE: points connected by edges



The Problem

The problem

The problem consists in the study of the connected components of this graph Γ_S for S *generic*.

It is not hard to see that, for generic values of S, the set S is itself a connected component which we call the *special component*.

The main combinatorial Theorem

Theorem

For generic choices of S the connected components of the graph Γ_S, different from the special component, are formed by

affinely independent points.

In particular each component has at most $n + 1$ points.

The proof is quite complex

it requires some algebraic geometry and a long combinatorial analysis.

General strategies:

$-\mathrm{i}u_t + u_{xx} + f(u, u_x, u_{xx}) = 0$, $-\mathrm{i}u_t + \Delta u - F(|u|^2)u = 0$,

- *generic* choice of tangential sites in order to avoid resonances due to $\lambda_j \in \mathbb{Z}$. Mostly algebraic/combinatoric methods (developed in [PP12]).
- Pseudo-differential calculus to deal with derivatives in the non-linearity (first developed for the forced Airy [BaldiBertiMontalto11])
- momentum conservation to deal with high multiplicity +Töplitz-Lipschitz property(developed in [EliassonKuksin] , [PXu]).

There is still a lot of active research on quasi-periodic solutions. Dealing with more general manifolds, quasi-linear PDEs on $\mathbb{T}^d$, studying stability and instability issues.

Drawbacks

- Quasi-periodic solutions are NOT typical even for integrable equations.
- The quasi-periodic solutions described are all at least C^∞ (both in time and space)
- In order to find finite regularity solutions one needs a non-linearity with finite regularity, and even in this case the regularity is very high.

To overcome these difficulties it is natural to look at almost-periodic solutions.

$$U : \mathbb{T}^{\mathbb{Z}} \quad \to \quad \mathcal{M} \subset \text{the phase space}$$

Very few results, most on 1d NLS or NLW with external parameters.

$$\mathrm{i}u_t - u_{xx} + V \star u + |u|^4 u = 0\,, \quad V \star u = \sum_{j\in\mathbb{Z}} V_j u_j e^{\mathrm{i}jx}$$

$iu_t - u_{xx} + V \star u + |u|^4 u = 0$

Even in this special setting one is only able to construct few almost-periodic solutions.

few $\rightsquigarrow$ special and/or very high regularity!

Authors	Decay of solution	Regularity of V
[Bo'96]	at least superexponential	analytic
[Pö'02]	at least superexponential	ℓ_2
[Geng-Xu'12–'16]	exponential	ℓ_2
[Bo'04 + below*]	subexponential	ℓ_∞
[BMP'22]	polynomial	ℓ_∞

below* = Cong-Liu-Shi-Yuan, Biasco-Massetti-P., Cong-Mi-Shi-Wu, Cong-Yuan (NLW),Cong, Cong-Wu high dimension

Finite regularity solutions for NLS

$$\mathrm{i}u_t + u_{xx} - V * u + F(|u|^2)u = 0 \qquad (4)$$

Theorem (Biasco-Massetti-P. 20)

For any $p > 1$ and for most choices of $V \in \ell^\infty$ there exist infinitely many (both weak and classical) almost periodic solutions

$$u(t,x) = \sum_j \hat{u}_j(t) e^{\mathrm{i}jx}\,, \quad \omega_j \sim j^2\,, \ \sup_j |\hat{u}_j| \langle j \rangle^p \ll 1\,.$$

Such solutions are approximately supported on sparse subsets of $\mathbb{Z}$.

For example Set $\mathcal{S} := \{j \in \mathbb{Z} : j = 2^h\,, \quad h \in \mathbb{N}\}$

we have a solution with $|\hat{u}_j| \sim \langle j \rangle^{-p}$ for all $j \in \mathcal{S}$

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Open problems

- Find Maximal tori (no conditions on $\mathcal{S}$) of finite regularity.
- NLS with multiplicative potential

$$\mathrm{i}u_t - u_{xx} + V(x)u + |u|^4 u = 0$$

- Degenerate KAM theory

$$u_{tt} + u_{xxxx} + mu + u^3 = 0$$

- Fix some (possibly very strong) regularity class, and prove that for most potentials $V \in \ell_\infty$ typical solutions in that class are almost-periodic.

Thanks for the attention!