Breaking heat kernel estimates into pieces

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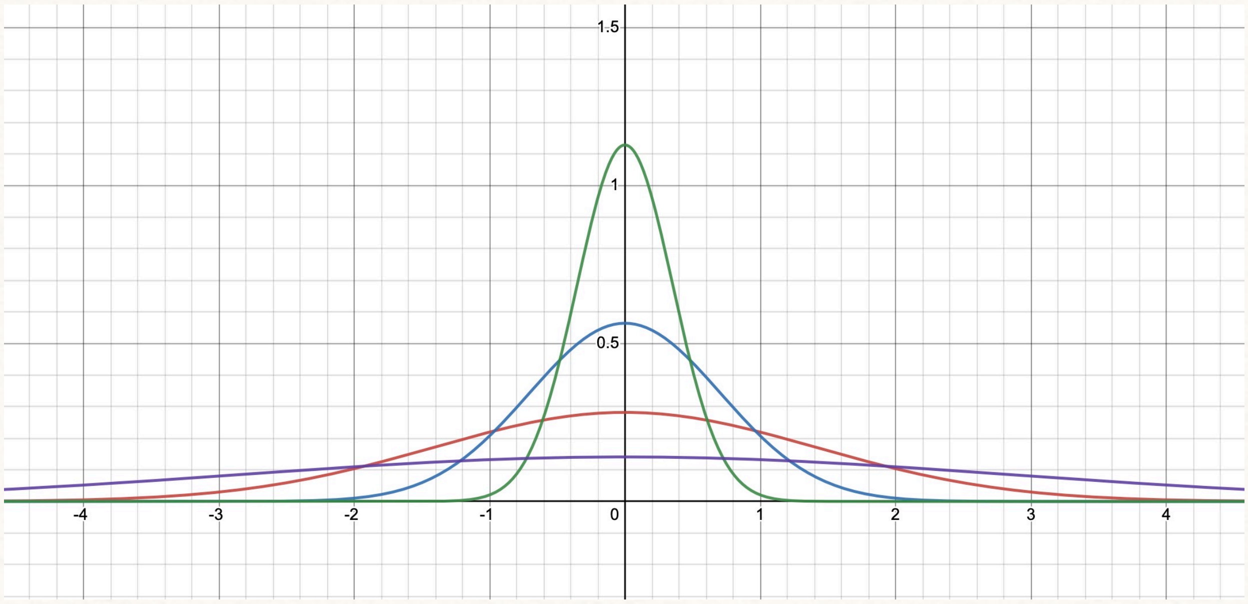
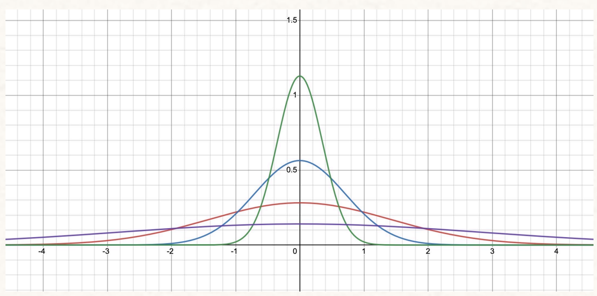


1. the heat kernel



$$(t,x,y) \longrightarrow (4\pi t)^{-n/2} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

The fundamental solution of the heat equation

$$(\partial_t - \Delta) u = 0 \quad \text{in} \quad \mathbb{R}^n$$

where $\Delta = \sum_1^n \partial_i^2$ $\qquad x = (x_1, \dots, x_n)$

Notation $p(t,x,y)$, $t \in (0,+\infty)$, $x,y \in M$

reference measure

$$\underbrace{e^{t\Delta}}_{\text{heat semigroup}} f(x) = \int_M \underbrace{p(t,x,y)}_{\text{heat kernel.}} f(y)\, d\mu(y)$$

- $M = \Omega$, connected domain in $\mathbb{R}^n$.

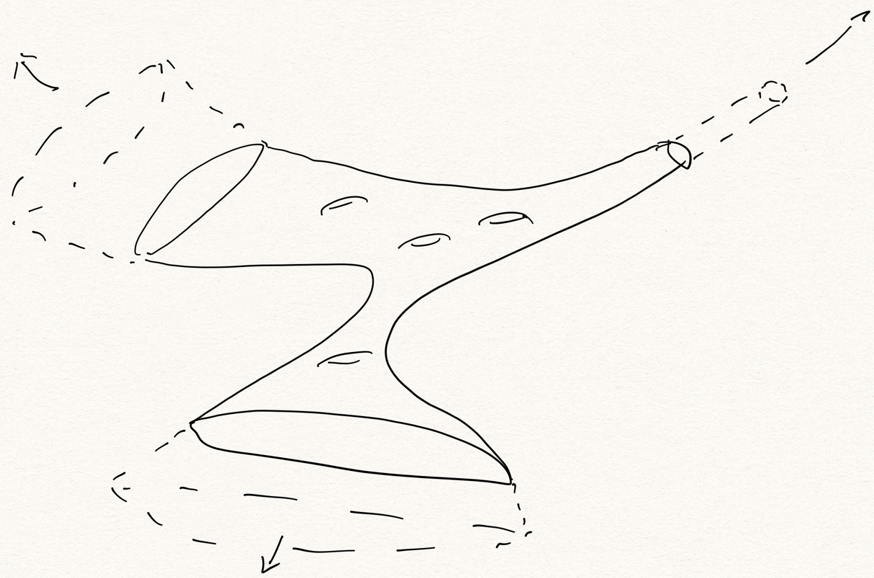
In a nice domain, Ω, one needs to specify some boundary condition :

- Neumann boundary condition
- Dirichlet boundary condition

- On a complete Riemannian manifold,

$$\Delta = \mathrm{div} \cdot \mathrm{grad}$$

is the Laplace-Beltrami operator.



- Weighted manifold:

  - Riemannian manifold (M, g)

    $\longrightarrow$ gradient, volume dv

  - weight $\sigma \in \mathcal{C}^\infty_+(M)$

    $$\longrightarrow \int |\mathrm{grad} f|^2 \sigma \, dv$$

    $$= - \int f \boxed{\Delta f} \sigma \, dv$$

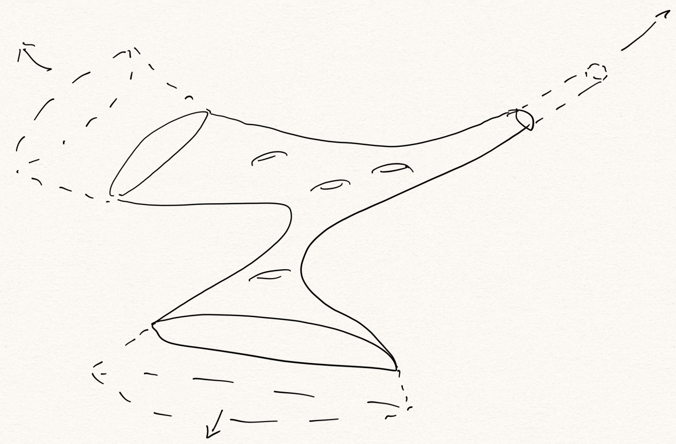
Bronian motion $(X_t)_{t>0}$

$$E_x(f(X_t)) = e^{t\Delta} f(x)$$
$$= \int_M p(t,x,y) f(y)\, dy$$

Bronian motion is reflected at the boundary
if Neumann condition

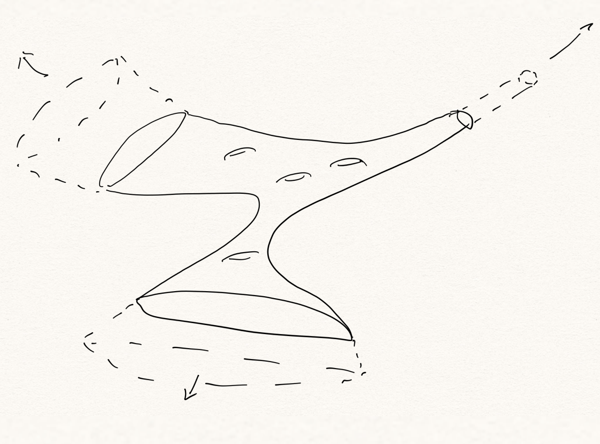
is killed at the boundary
if Dirichlet condition

What is our goal?



Two-sided estimates of $p(t,x,y)$ holding for all $t>0$ and $x,y\in M$.

- For any ball B, be able to estimate
$$\mathbb{P}_x(X_t \in B) = \int_B p(t,x,y)\,dy .$$
- Given x, estimate $t \to \max_y \{p(t,x,y)\}$ and find the location of those y where this maximum is attained



II Main principle:

"Breaking into pieces" & "Gluing back together"

Step 1: Identifying "pieces" whose own heat kernels can be understood.

Step 2: Estimate the heat kernel of the orignal manifold by "gluing back together" our understanding of the pieces.

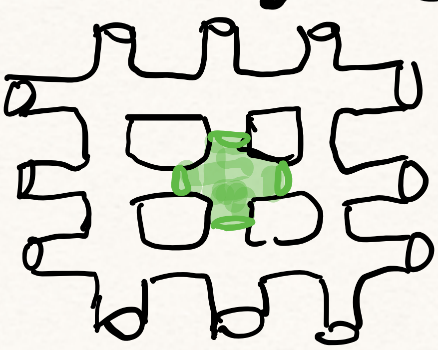
this requires understanding the interactions between the pieces.

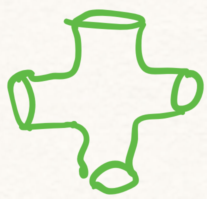
Examples:

- "1 piece": we need to know how to study the heat kernel!

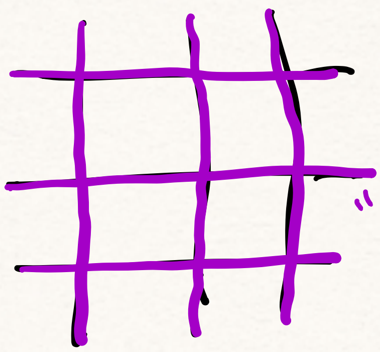
- "pieces = points": heat kernel on each "piece" is trivial. Understanding the "interactions" is hard (same as first bullet)

- "discretisation"



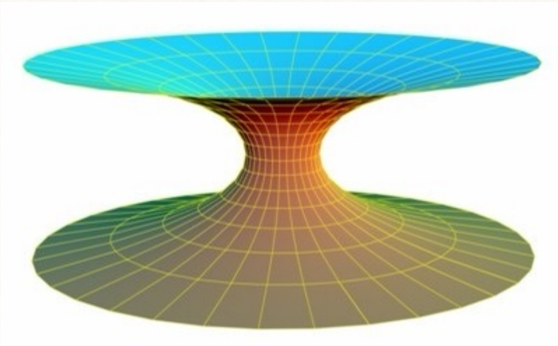


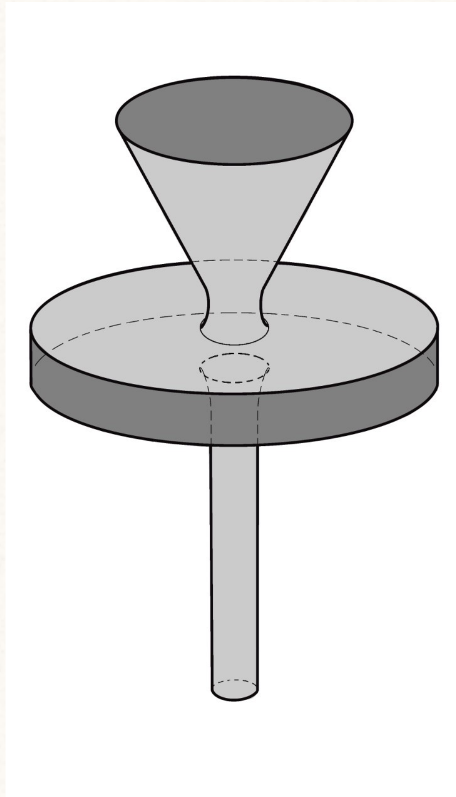
"Pieces"



"interactions"

Main examples for today:





III What is a good piece?

"Harnack manifolds"

- Volume doubling: $\mu(B(x,2r)) \leq D\mu(B(x,r))$
  uniform at all scales and locations

- Poincaré inequality
$$\int_B |f - f_B|^2 d\mu \leq P r^2 \int_B |\nabla f|^2 d\mu$$
uniformly for all balls $B = B(x,r)$.

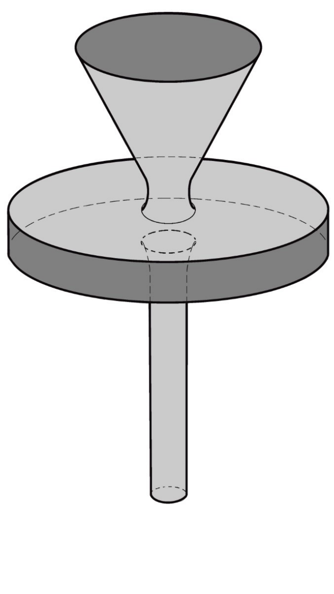
Doubling & Poincaré ?

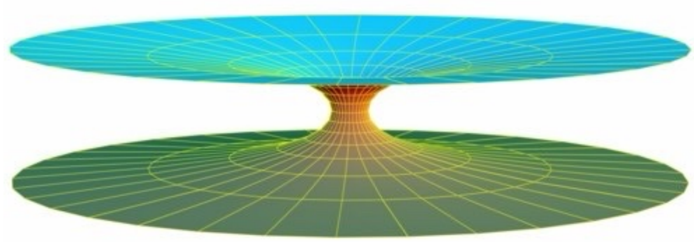
$\mathbb{R}^n$

Convex sets in $\mathbb{R}^n$

Non-negative Ricci curvature.

- Peter Buser (Poincaré)
- J. Cheeger, M. Gromov, M. Taylor (Doubling)
  (Bishop-Gromov comparison theorem).



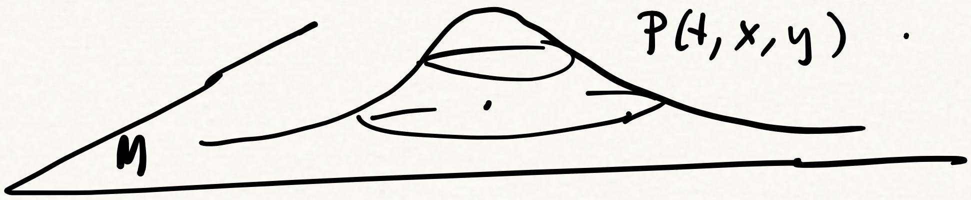


Theorem (Independently, A. Grigor'yan; L.S.C.)

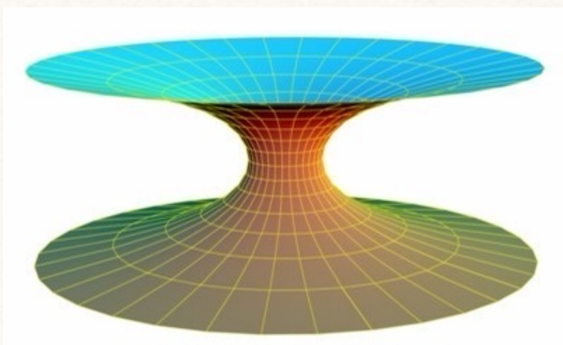
Volume Doubling & Poincaré

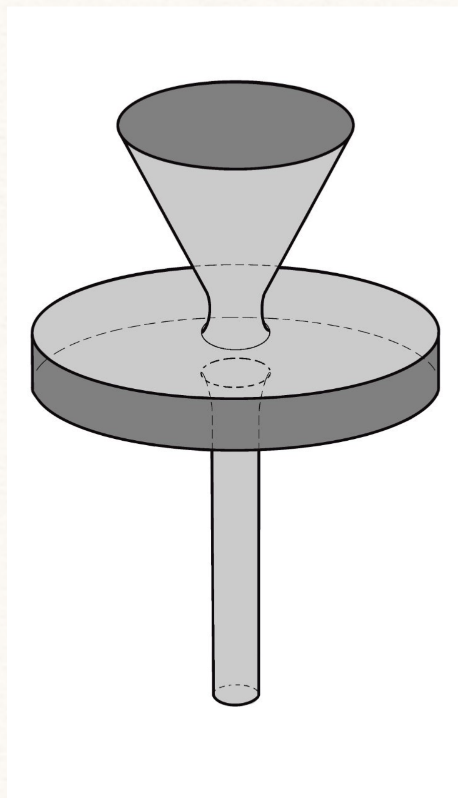
$$\Updownarrow$$

$$\frac{c_3}{\mu(B(x,\sqrt{t}))}\exp\left(-c_4\frac{d(x,y)^2}{t}\right) \le p(t,x,y) \le \frac{c_1}{\mu(B(x,\sqrt{t}))}\exp\left(-c_2\frac{d(x,y)^2}{t}\right)$$



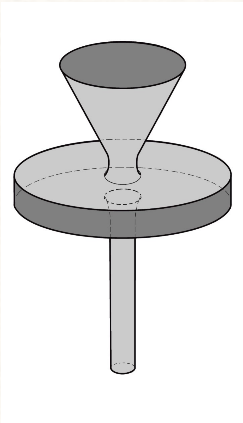
A "good piece" is a Harnack manifold.

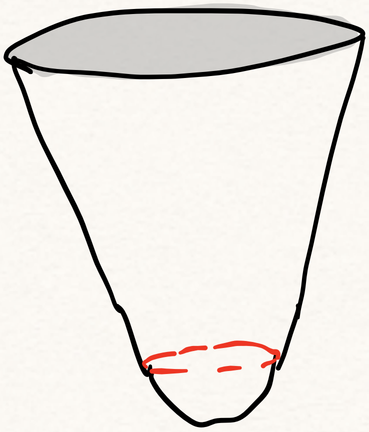




IV What do we need to know about "good pieces"?

① Each piece is a Harnack manifold.





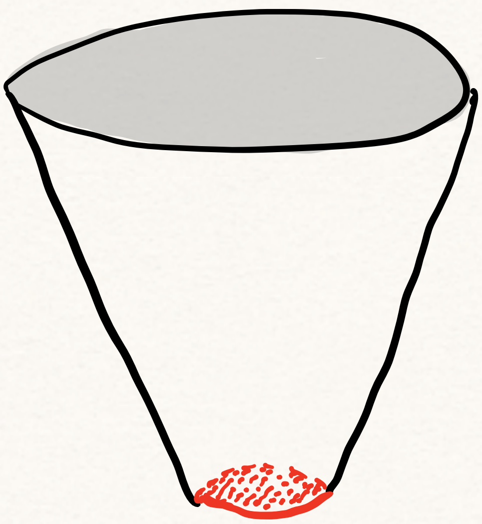
"Harnack manifold"

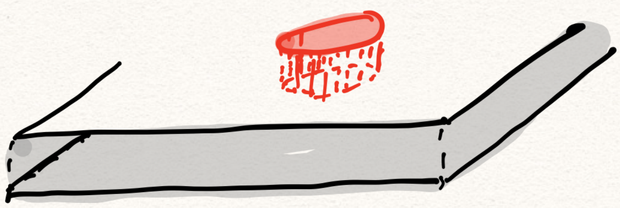
$\Updownarrow$

Good heat kernel

$$\frac{1}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{d(x,y)^2}{t}\right)$$

② Hitting probability $\mathbb{P}_x(\tau < t)$

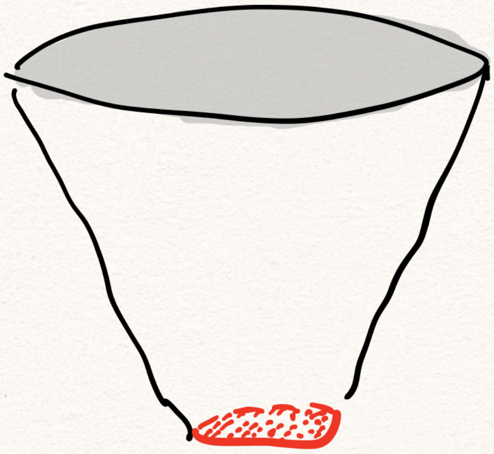




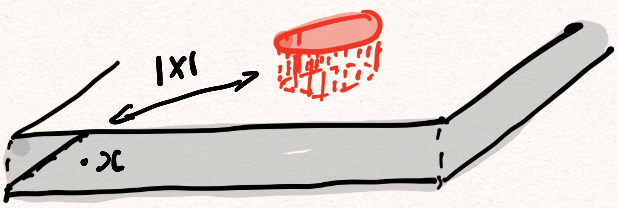
τ is the first hitting time of the "cut".

Harmonic profile: $h > 0$ $\quad \Delta h = 0$

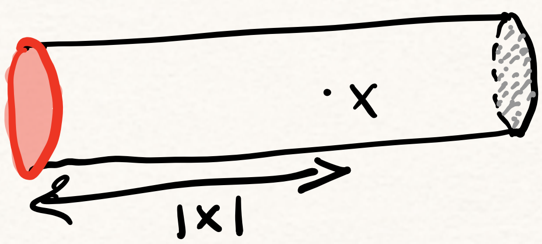
$h \equiv 0$ on red boundary



$h \asymp 1$ at infinity

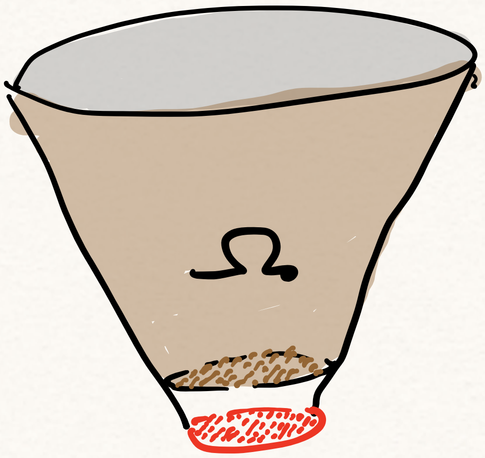


$h(x) \asymp \log(1+|x|)$

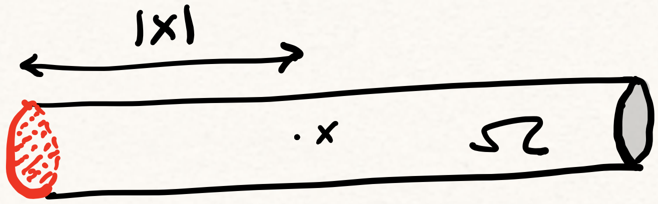


$h(x) \asymp |x|$

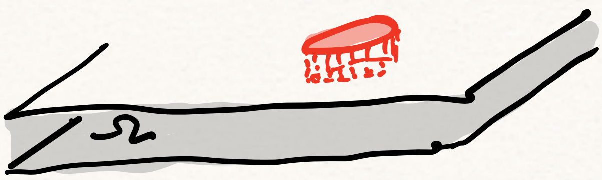
③ The heat kernel with 0 boundary condition.



$$P_\Omega(t,x,y) \asymp \frac{1}{\mu(B(x,\sqrt{t}))}\exp\left(-\frac{d(x,y)^2}{t}\right) \asymp P(t,x,y)$$

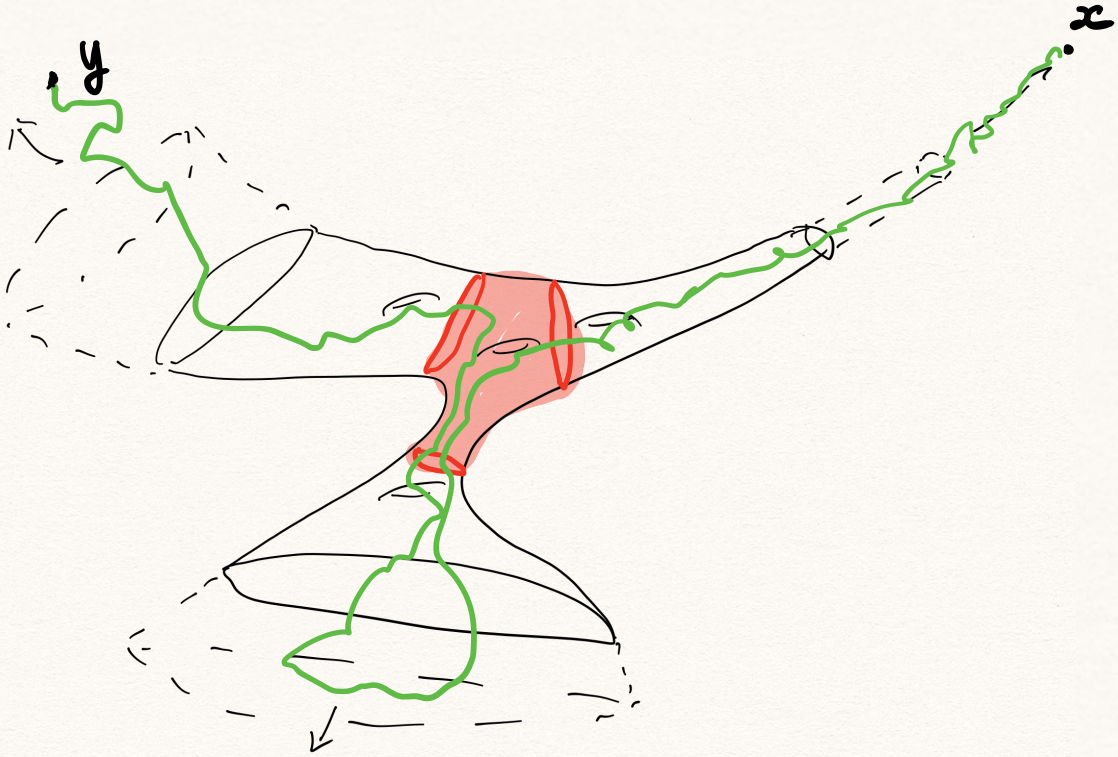


$$P_\Omega(t,x,y) \asymp \frac{|x||y|}{\sqrt{t}\,(\sqrt{t}+|x|)(\sqrt{t}+|y|)}\exp\left(-\frac{d(x,y)^2}{t}\right)$$



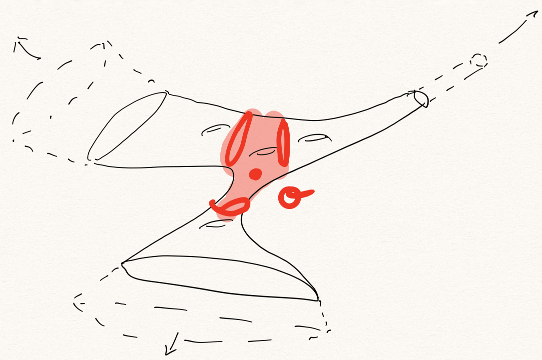
$$P_\Omega(t,x,y) \asymp \frac{\log(1+|x|)\log(1+|y|)}{t\,(\sqrt{t}+\log^2(1+|x|))(\sqrt{t}+\log^2(1+|y|))}\exp\left(-\frac{d(x,y)^2}{t}\right)$$

V Gluing back pieces together



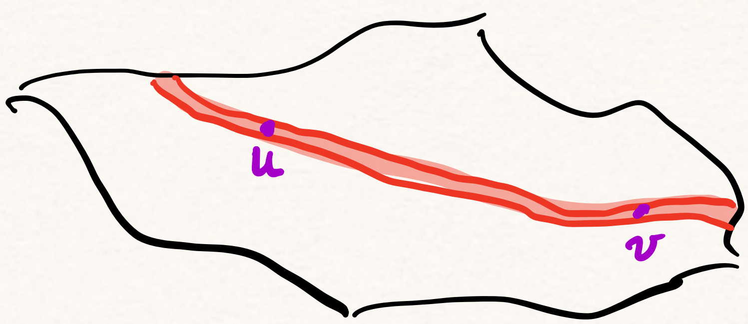
- Finitely many good pieces connected through a compact central part

$p(t, o, o) \simeq$?



- More generally

$p(t, u, v) \simeq$?



VI What is the "final product" of these investigations?

global two-sided estimates of the heat kernel

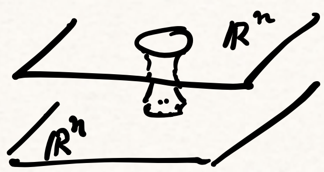
$$q(c_1, t, x, y) \leq p(t, x, y) \leq q(c_2, t, x, y)$$

q is a function of t, x, y and some constants
it depends on x, y through simple geometric quantities including volume, distance $d(x,y)$ distances of x and y to the cut, which pieces do x and y belong to....

<u>Example 1</u>

$\mathbb{R}^n \# \mathbb{R}^n$

<u>$n > 2$</u>



$$q(c, t, x, y) = \frac{1}{c\, t^{n/2}} \left(\frac{1}{\|x\|^{n-2}} + \frac{1}{\|y\|^{n-2}} \right) \exp\left(- \frac{c\, d_+(x,y)}{t} \right)$$

$$+ \frac{1}{c\, t^{n/2}} \exp\left(- \frac{c\, d_\emptyset(x,y)}{t} \right)$$

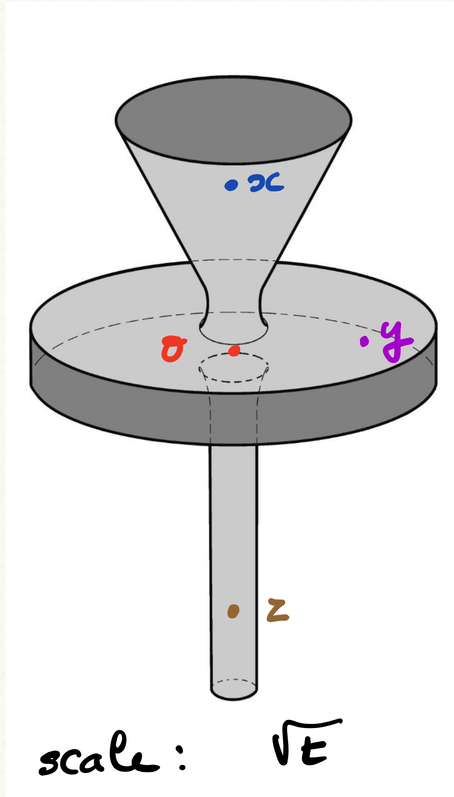
Example 2 $\quad (t \geq 1)$

$p(t,o,o) \asymp \dfrac{1}{t \log^2(1+t)}$

$p(t,o,x) \asymp \dfrac{1}{t^{3/2}}$

$p(t,o,y) \asymp \dfrac{1}{t \log(1+t)}$

$p(t,o,z) \asymp \dfrac{1}{t}$



- Example 1, $\mathbb{R}^n \# \mathbb{R}^n$, $n>2$ is representative of the case when several Harnack manifolds that are non-parabolic are glued together through a compact central piece.

$$p(t,o,o) \asymp \frac{1}{V_{min}(\sqrt{t})}$$

- Example 2 is representative of the case when at least one "good piece" is non-parabolic.

- the catenoid is representative of the case when all the "good pieces" are parabolic. It is the most difficult case and remained unfinished.

Conclusions :

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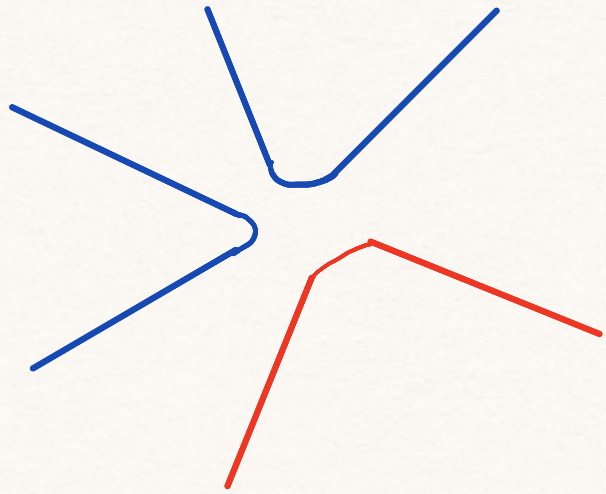

• Cutting and gluing along non-compact locations to buid a robust widely applicable theory -

Thank you!