On hydrodynamic limits of fractional PDEs from stochastic interacting particle systems

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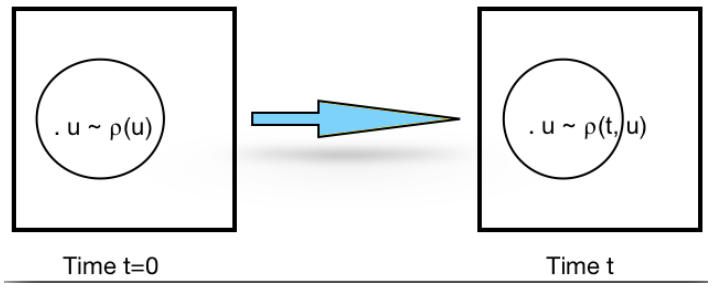


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Collaborators:



Physical motivation:



- Find the invariant states of a system.
- Characterize them by a quantity $\varrho(\cdot)$.
- Fix $u \in \Lambda$ and a neighbourhood Λ_u (microscopically big). Due to interaction, the system reaches an equilibrium $\varrho(u)$.
- Let time evolve. Now the equilibrium close to u is given by $\varrho(t, u)$. How does $\varrho(t, u)$ evolve?

- We discretize a volume Λ according to a parameter N and get Λ_N.
- Two scales for space/time.
- In each cell we put a random number of particles.
- The dynamics conserves some quantity of interest.
- Waiting times are given by independent Poisson processes; particle system is a Markov chain - *loss of memory*.

QUESTION: What is the macroscopic law describing the evolution of the conserved quantity of the system?

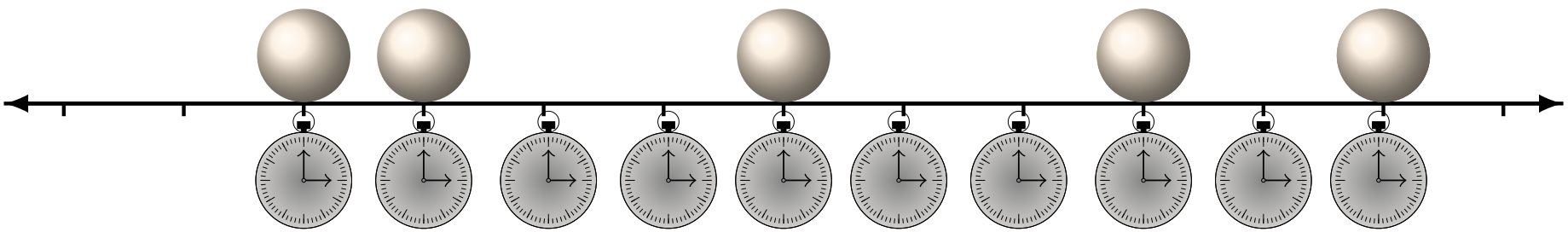
SSEP:

- It is well known that for SSEP one can get the heat equation, either on torus, interval, and also with boundary conditions.
- The aim of this talk is to overview:
  - the derivation of **fractional** PDEs with several boundary conditions in different domains, replacing the laplacian operator by the regional fractional laplacian operator;
  - analyse how local/non-local defects at micro level propagate to the macro level.

Exclusion process:

- N: scaling parameter;
- **Space**:
  - microscopic (discrete);
  - macroscopic (continuous);
- **Time**:
  - microscopic $t\theta(N)$;
  - macroscopic t;
- Independent Poissonian clocks;
- Transition probability $p(\cdot)$;
- $\eta_t^N(x) =$ number of particles at site x;
- **Markov processes**; (continuous time)
- Density $\sum_x \eta_t^N(x)$ is conserved.

Exclusion: After one ring of a clock a particle jumps from x to y at rate $p(y-x)$.

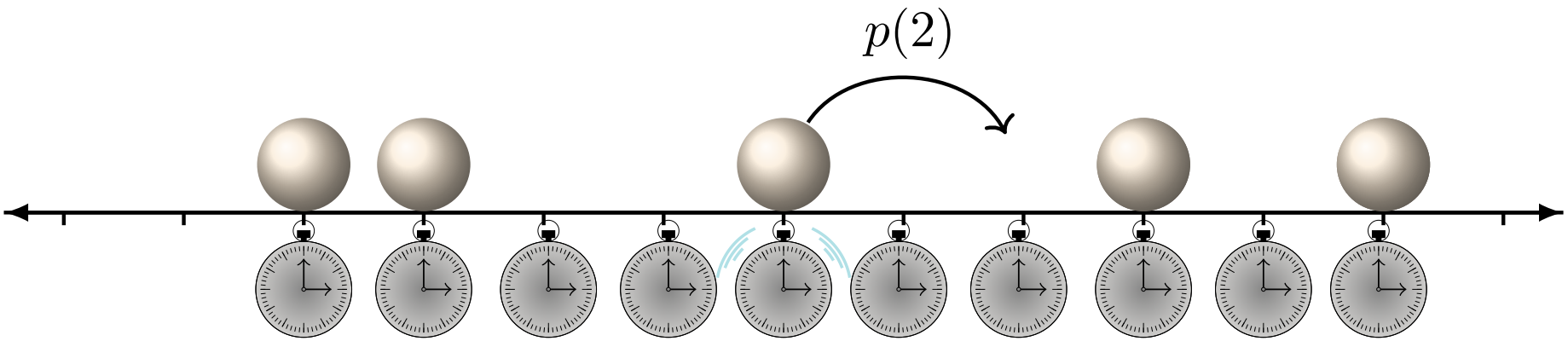


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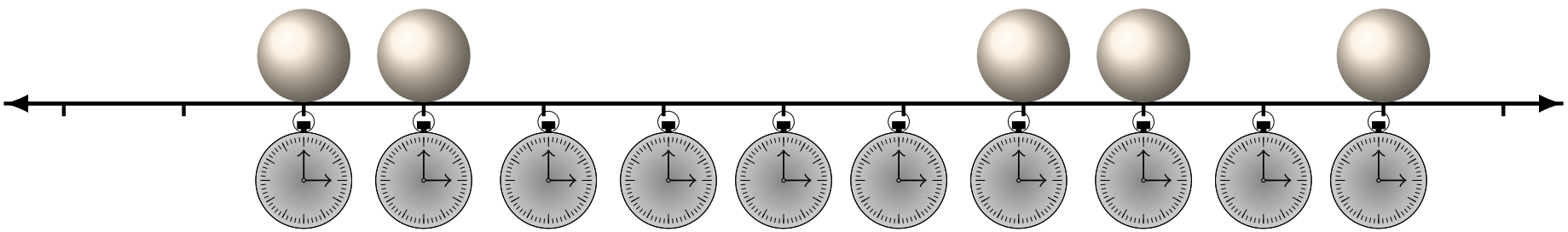


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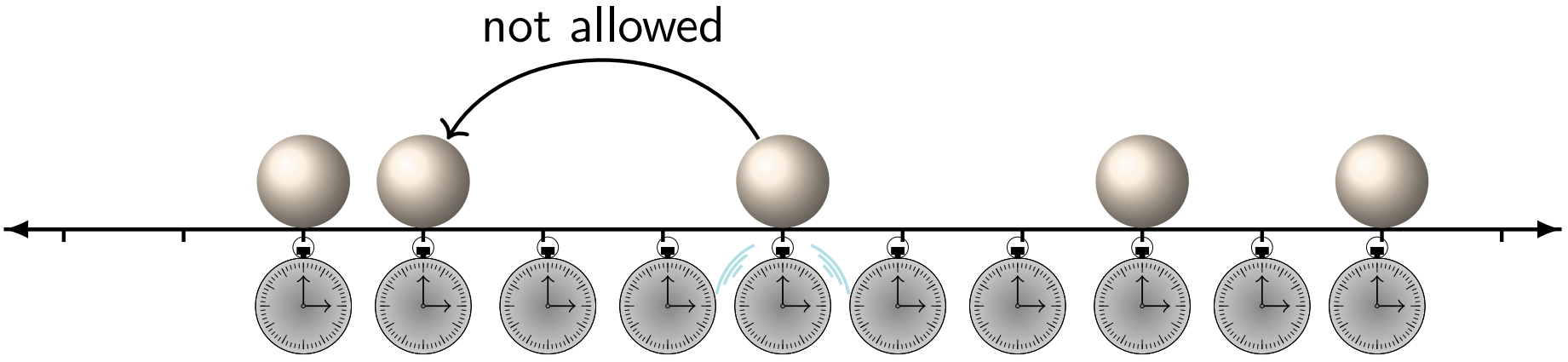


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Exclusion: the forbidden jumps.

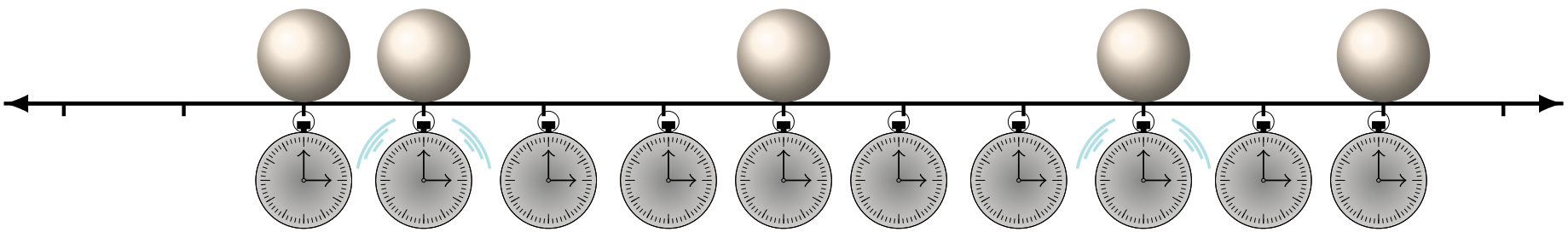


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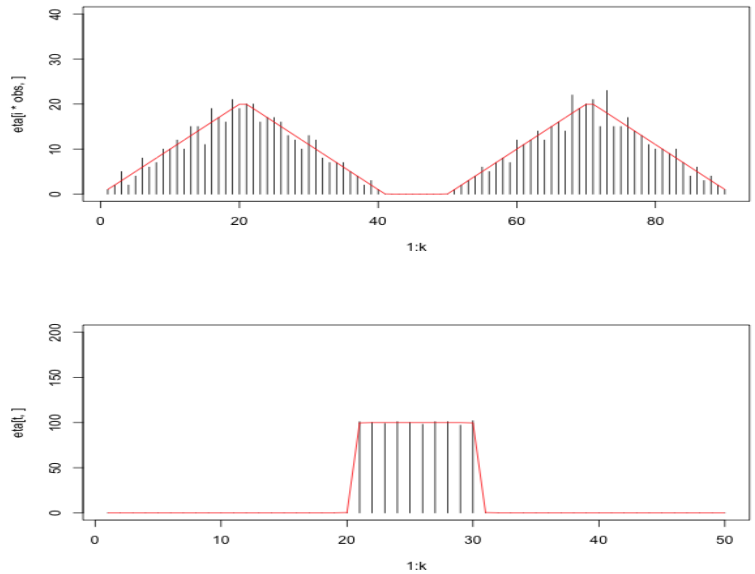
This cannot happen.

- Independent Poissonian clocks;
- Transition probability $p(\cdot)$;
- $\eta_t^N(x) =$ number of particles at site x;
- **Markov processes**; (continuous time)
- Density $\sum_x \eta_t^N(x)$ is conserved.



Hydrodynamic Limit: empirical measure

Goal:
$\pi_t^N(\eta, du) \to_{N\to+\infty} \varrho_t(u)du,$
where
$\pi_t^N(\eta, du) = \frac{1}{N}\sum_x \eta_t^N(x)\delta_{\frac{x}{N}}(du)$
and $\varrho_t(\cdot)$ is solution of a PDE.



Possible hydrodynamic equations

Heat Equation (HE): $\partial_t \varrho_t = \Delta \varrho_t$ (p symmetric, tN^2)
Porous Medium (PM): $\partial_t \varrho_t = \Delta \varrho_t^m, 2 \le m \in \mathbb{N}$ (p symm., tN^2)
Fractional HE: $\partial_t \varrho_t = -(-\Delta)^{\gamma/2} \varrho_t$ ($p(\cdot) = \frac{c}{|\cdot|^{1+\gamma}}$, tN^γ, $\gamma \in (0,2)$)
Fract. PM: $\partial_t \varrho_t = -(-\Delta)^{\gamma/2} \varrho_t^2$ ($p(\cdot) = \frac{c}{|\cdot|^{1+\gamma}}$, tN^γ, $\gamma \in (0,2)$)

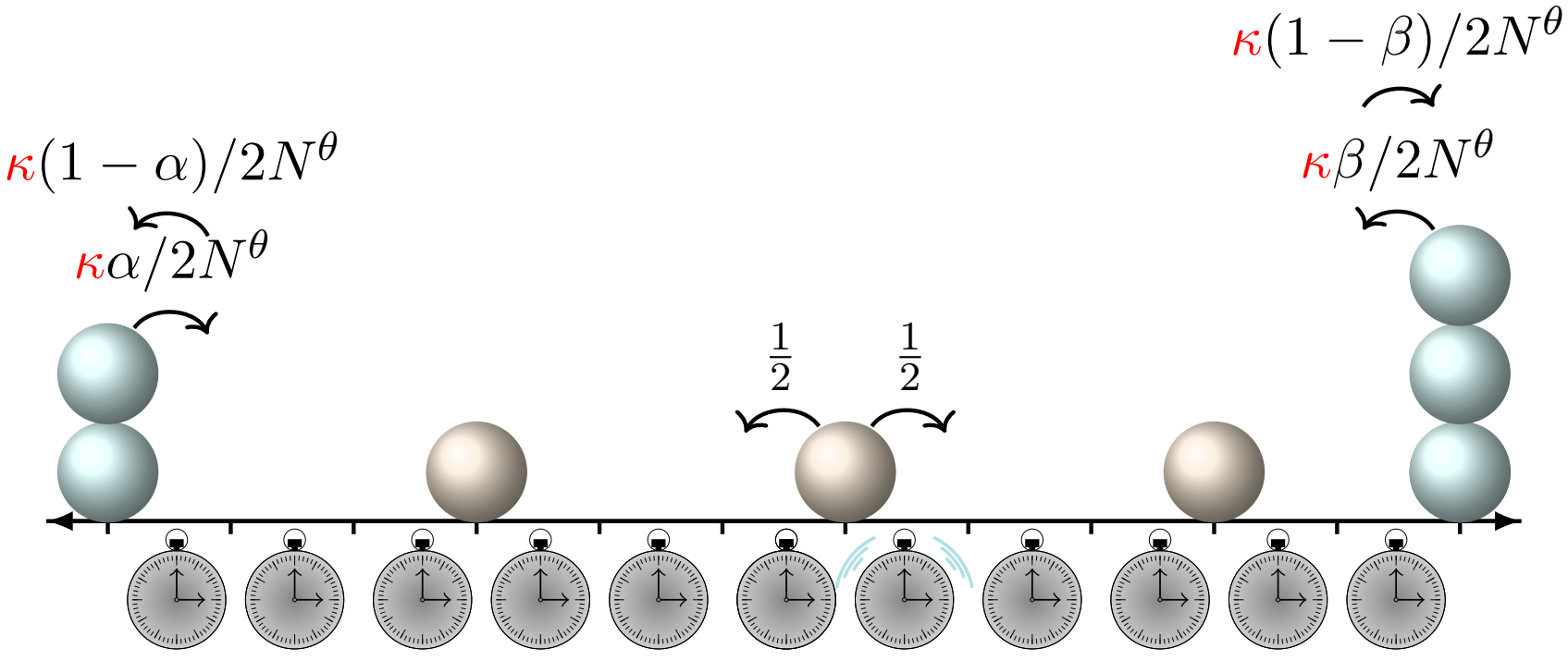
The focus of this presentation:

- I will present the hydrodynamic limit for an exclusion process in contact with stochastic reservoirs when jumps are long range given by a symmetric probability transition rate:
  - with finite variance;
  - with infinite variance.

Let us start with the simplest case: jumps to nearest-neighbors.

Now $\Lambda = [0,1]$ and $\Lambda_N = \{1, ..., N-1\}$.
The state space of the Markov process η_t^N is $\Omega_N = \{0,1\}^{\Lambda_N}$.

SSEP in contact with reservoirs:



Invariant measures:

- **If $\alpha = \beta = \varrho$ the Bernoulli product measures are invariant (equilibrium measures):** $\nu_\varrho(\eta : \eta(x) = 1) = \varrho$.

- **If $\alpha \neq \beta$ Bernoulli product measures are not invariant, but since we have a finite state irreducible Markov process there exists a UNIQUE invariant measure: the non-equilibrium stationary state - NESS denoted by μ_{ss}.**

- **By the matrix ansatz method[1] one can get information about this measure.** (Not in the long jumps case.)

[1] Derrida, B., Evans, M., Hakim, V., Pasquier, V.: Exact solution of a 1D asymmetric exclusion model using a matrix formulation. Journal of Physics A: Mathematical and General 26 no. 7, 1493, (1993).

Hydrodynamic Limit:

♣ For $\eta \in \Omega_N$, let $\pi_t^N(\eta, dq) = \frac{1}{N-1} \sum_{x=1}^{N-1} \eta_{tN^2}(x) \delta_{x/N}(dq)$, be the *empirical measure. (Diffusive time scaling!)*

♣ Assumption: fix $g : [0,1] \to [0,1]$ measurable and probability measures $\{\mu_N\}_{N \geq 1}$ such that for every $H \in C([0,1])$,

$$\frac{1}{N-1} \sum_{x=1}^{N-1} H(\tfrac{x}{N})\, \eta(x) \to_{N \to +\infty} \int_0^1 H(q)\, g(q) dq,$$

wrt μ_N. (μ_N is associated to $g(\cdot)$)

♣ Then: for any $t > 0$,

$$\pi_t^N(\eta, dq) \to_{N \to +\infty} \varrho(t,q) dq,$$

wrt $\mu_N(t)$, where $\varrho(t,q)$ evolves according to a PDE, the hydrodynamic equation.

Hydrodynamic Limit:

Theorem [Baldasso el al.]

Let $g : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated to $g(\cdot)$. Then, for any $0 \leq t \leq T$,

$$\mathbb{P}_{\mu_N}\left(\left|\tfrac{1}{N-1}\sum_{x=1}^{N-1} H(\tfrac{x}{N})\eta_{tN^2}(x) - \int_0^1 H(q)\varrho(t,q)dq\right| > \delta\right) \to_N 0$$

and $\varrho_t(\cdot)$ is the UNIQUE weak solution of the heat equation $\partial_t \varrho_t(q) = \frac{1}{2}\partial_q^2 \varrho_t(q)$ with initial condition $g(\cdot)$ and for $\theta > 1$ **Neumann b.c.** $\partial_q \varrho_t(0) = \partial_q \varrho_t(1) = 0$, for $\theta = 1$ **Robin b.c.** $\partial_q \varrho_t(0) = \kappa(\varrho_t(0) - \alpha)$, $\partial_q \varrho_t(1) = \kappa(\beta - \varrho_t(1))$ and for $\theta < 1$ **Dirichlet b.c.** $\varrho_t(0) = \alpha$, $\varrho_t(1) = \beta$.

The result on the torus can be seen in Kipnis, C., Landim, C., Scaling Limits of Interacting particle systems, Springer-Verlag and on the torus with a slow bond in Franco, T., G.P., Neumann, A., Phase transition for the Heat equation with Robin's boundary conditions and exclusion process, Trans. Amer. Math. Soc., 367 (9), 6131–6158, (2015).

Hydrostatic Limit:



Theorem: Let μ_{ss} be the stationary measure for the process $\{\eta_{tN^2}\}_{t\geq 0}$. Then, μ_{ss} is associated to $\bar{\varrho} : [0,1] \to [0,1]$ given on $q \in (0,1)$ by

$$\bar{\varrho}(q) = \begin{cases} (\beta-\alpha)q + \alpha \,;\, \theta < 1, \\ \frac{\kappa(\beta-\alpha)}{2+\kappa} q + \alpha + \frac{\beta-\alpha}{2+\kappa} \,;\, \theta = 1, \\ \frac{\beta+\alpha}{2} \,;\, \theta > 1, \end{cases}$$

$\bar{\varrho}(\cdot)$ is a stationary solution of the hydrodynamic equation.

The proof:

Goes through the entropy method [2].

- Tightness of $\mathbb{Q}_N$, where $\mathbb{Q}_N$ is induced by $\mathbb{P}_{\mu_N}$ and the map

$$\pi_\cdot^N : \mathcal{D}([0,T],\Omega_N) \longrightarrow \mathcal{D}([0,T],\mathcal{M}_+)$$

- Characterization of limit points: concentrated on trajectories of measures absolutely continuous wrt Lebesgue whose density is a weak solution of the PDE:

$$\mathbb{Q}(\pi_\cdot : \pi_t(dq) = \varrho(t,q)dq \text{ and } \varrho_t(q) \text{ is solution to the PDE}) = 1.$$

[2] Guo, M.Z., Papanicolaou, G.C., Varadhan, S.R.S.: Nonlinear diffusion limit for a system with nearest neighbor interactions. Commun. Math. Phys. 118, 31–59, (1988).

The notion of weak solution:

Let $g : [0,1] \to [0,1]$ be measurable. $\varrho : [0,T] \times [0,1] \to [0,1]$ is a weak solution to HE with Dirichlet b.c. if:

- $\varrho \in L^2(0,T;\mathcal{H}^1)$;
- ϱ satisfies the weak formulation:

$$\int_0^1 \varrho_t(q)H_t(q) - g(q)H_0(q)\,dq - \int_0^t \int_0^1 \varrho_s(q)\Big(\tfrac{1}{2}\partial_q^2 + \partial_s\Big)H_s(q)\,ds\,dq$$
$$+ \tfrac{1}{2}\int_0^t \beta\partial_q H_s(1) - \alpha\partial_q H_s(0)\,ds = 0,$$

for all $t \in [0,T]$ and any function $H \in C_0^{1,2}([0,T]\times(0,1))$.

Definition

The Sobolev space $\mathcal{H}^1$ on $(0,1)$ is the Hilbert space defined as the completion of $C^\infty([0,1])$ for the norm $\|\cdot\|_{\mathcal{H}^1}^2 := \|\cdot\|_2^2 + \|\cdot\|_1^2$, where $\|H\|_1^2 = \int_0^1 (\partial_q H(q))^2\,dq$, The space $L^2(0,T;\mathcal{H}^1)$ is the set of measurable functions $f : [0,T] \to \mathcal{H}^1$ such that $\int_0^T \|f_s\|_{\mathcal{H}^1}^2\,ds < \infty$.

Other notion of solution:

Let $g : [0,1] \to [0,1]$ be measurable. $\varrho : [0,T] \times [0,1] \to [0,1]$ is a weak solution to HE with Dirichlet b.c. if:

- $\varrho \in L^2(0,T;\mathcal{H}^1)$;
- ϱ satisfies the weak formulation:

$$\int_0^1 \varrho_t(q)H_t(q) - g(q)H_0(q)\,dq$$
$$- \int_0^t \int_0^1 \varrho_s(q)\Big(\tfrac{1}{2}\partial_q^2 + \partial_s\Big)H_s(q)\,ds\,dq = 0,$$

  for all $t \in [0,T]$ and any function $H \in C_c^{1,2}([0,T] \times (0,1))$;
- $\varrho_t(0) = \alpha$ and $\varrho_t(1) = \beta$, for almost every $t \in (0,T]$.

How do we formulate the solution:

A simple computation shows that

$$\begin{aligned} N^2\mathscr{L}_N\langle \pi_s^N, H\rangle &= \langle \pi_s^N, \tfrac{1}{2}\Delta_N H\rangle \\ &+ \tfrac{1}{2}\nabla_N^+ H(0)\eta_{sN^2}(1) - \tfrac{1}{2}\nabla_N^- H(1)\eta_{sN^2}(N-1) \\ &+ \tfrac{\kappa}{2} N^{1-\theta} H\Big(\tfrac{1}{N}\Big)(\alpha - \eta_{sN^2}(1)) \\ &+ \tfrac{\kappa}{2} N^{1-\theta} H\Big(\tfrac{N-1}{N}\Big)(\beta - \eta_{sN^2}(N-1)) \end{aligned}$$

If $H(0) = H(1) = 0$, then from Dynkin's formula, we get

$$\begin{aligned} M_t^N(H) &= \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \tfrac{1}{2}\Delta_N H\rangle ds \\ &- \tfrac{1}{2}\int_0^t \nabla_N^+ H(0)\eta_{sN^2}(1) - \nabla_N^- H(1)\eta_{sN^2}(N-1) ds + O(N^{-\theta}). \end{aligned}$$

How do we formulate the solution $\theta \in (0,1)$:

Replacing $\eta_{sN^2}(1)$ by α and $\eta_{sN^2}(N-1)$ by β ($\theta < 1$!) then

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \tfrac{1}{2}\Delta_N H\rangle ds$$
$$- \tfrac{1}{2}\int_0^t \nabla_N^+ H(0)\alpha - \nabla_N^- H(1)\beta ds + O(N^{-\theta}).$$

Take the expectation and assume that
$\varrho_t^N(x) = \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)] \sim \varrho_t(x/N)$, for N big, to get

$$\int_0^1 \varrho_t(q)H(q) - \varrho_0(q)H(q)dq - \int_0^t \int_0^1 \tfrac{1}{2}\partial_q^2 H(q)\varrho_s(q)dqds$$
$$- \tfrac{1}{2}\int_0^t \partial_q H(0)\alpha - \partial_q H(1)\beta ds = 0.$$

How do we formulate the solution $\theta \leq 0$:

Replacing $\eta_{sN^2}(1)$ by α and $\eta_{sN^2}(N-1)$ by β ($\theta < 1$!) then

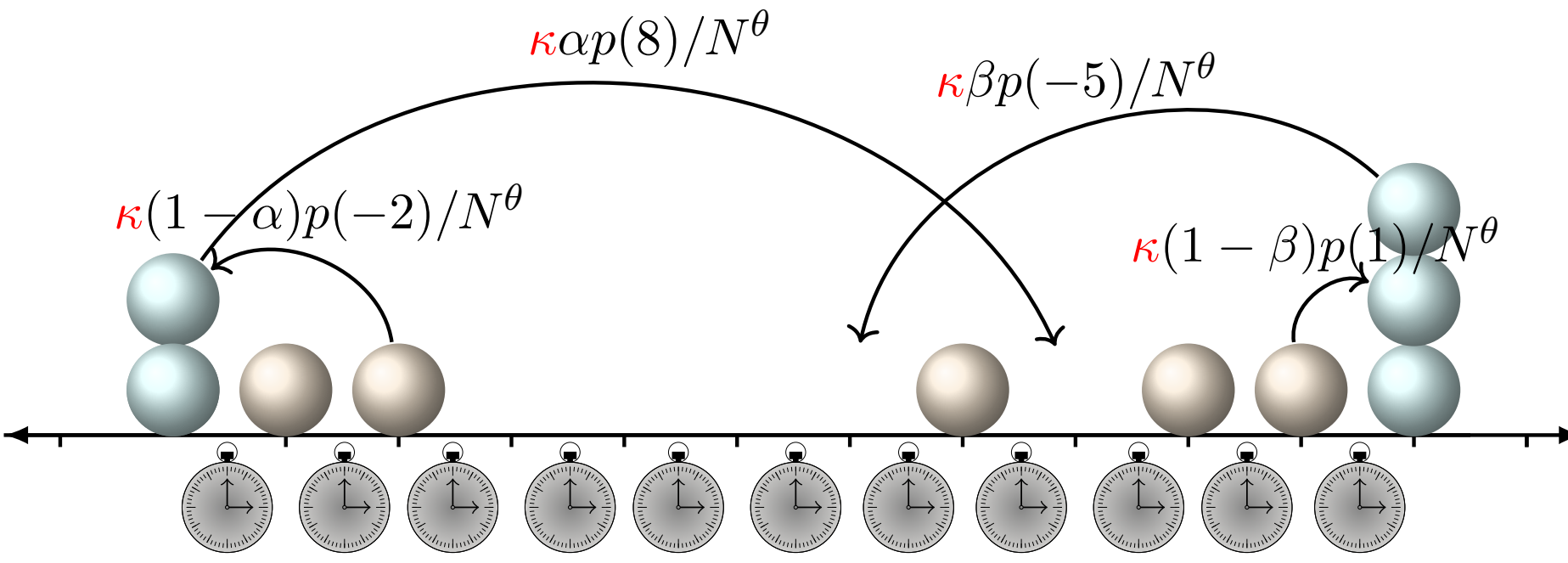
$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \tfrac{1}{2}\Delta_N H\rangle ds$$
$$- \tfrac{1}{2}\int_0^t \nabla_N^+ H(0)\alpha - \nabla_N^- H(1)\beta ds + O(N^{-\theta}).$$

The error above blows up since $\theta < 0$. Fix $H \in C_c^2((0,1))$. Take the expectation and assume that $\varrho_t^N(x) = \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)] \sim \varrho_t(x/N)$, for N big, to get

$$\int_0^1 \varrho_t(q)H(q) - \varrho_0(q)H(q)dq - \int_0^t \int_0^1 \tfrac{1}{2}\partial_q^2 H(q)\varrho_s(q)dqds = 0.$$

The space of test functions is too little, we need extra conditions for uniqueness: $\varrho_t(0) = \alpha$ and $\varrho_t(1) = \beta$, a.e. $t \in (0,T]$.

Exclusion in contact with infinitely many reservoirs



What if jumps are arbitrarily big?

Let $p(\cdot)$ be a translation invariant transition probability given at $z \in \mathbb{Z}$ by

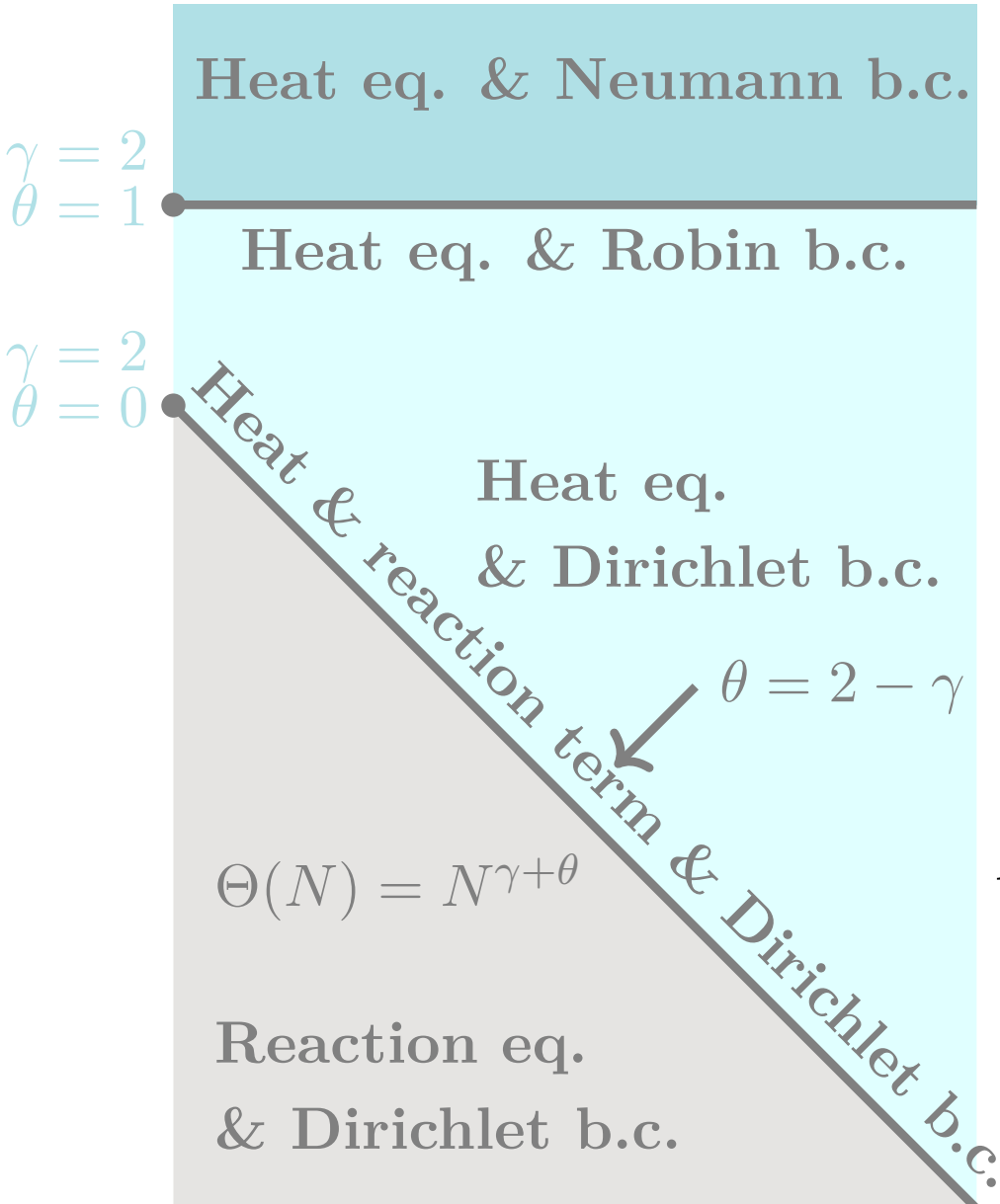
$$p(z) = \begin{cases} \dfrac{c_\gamma}{|z|^{\gamma+1}}, \ z \neq 0, \\ 0, \ z = 0, \end{cases}$$

where c_γ is a normalizing constant. Since $p(\cdot)$ is symmetric it is mean zero, that is:

$$\sum_{z \in \mathbb{Z}} z p(z) = 0$$

and take (by now) $\gamma > 2$ so that we define its variance by

$$\sigma_\gamma^2 = \sum_{z \in \mathbb{Z}} z^2 p(z) < \infty.$$



- **Heat equation:**

$$\partial_t \varrho_t(q) = \frac{\sigma^2}{2} \partial_q^2 \varrho_t(q)$$

  - $\theta = 1$ **Robin b.c.:**

$$\partial_q \varrho_t(0) = \frac{2m\kappa}{\sigma^2}(\varrho_t(0) - \alpha),$$
$$\partial_q \varrho_t(1) = \frac{2m\kappa}{\sigma^2}(\beta - \varrho_t(1)),$$

- **Reaction-diffusion eq.:**

$$\partial_t \varrho_t(q) = \frac{\sigma^2}{2} \partial_q^2 \varrho_t(q) + \kappa(V_0(q) - V_1(q)\varrho_t(q))$$

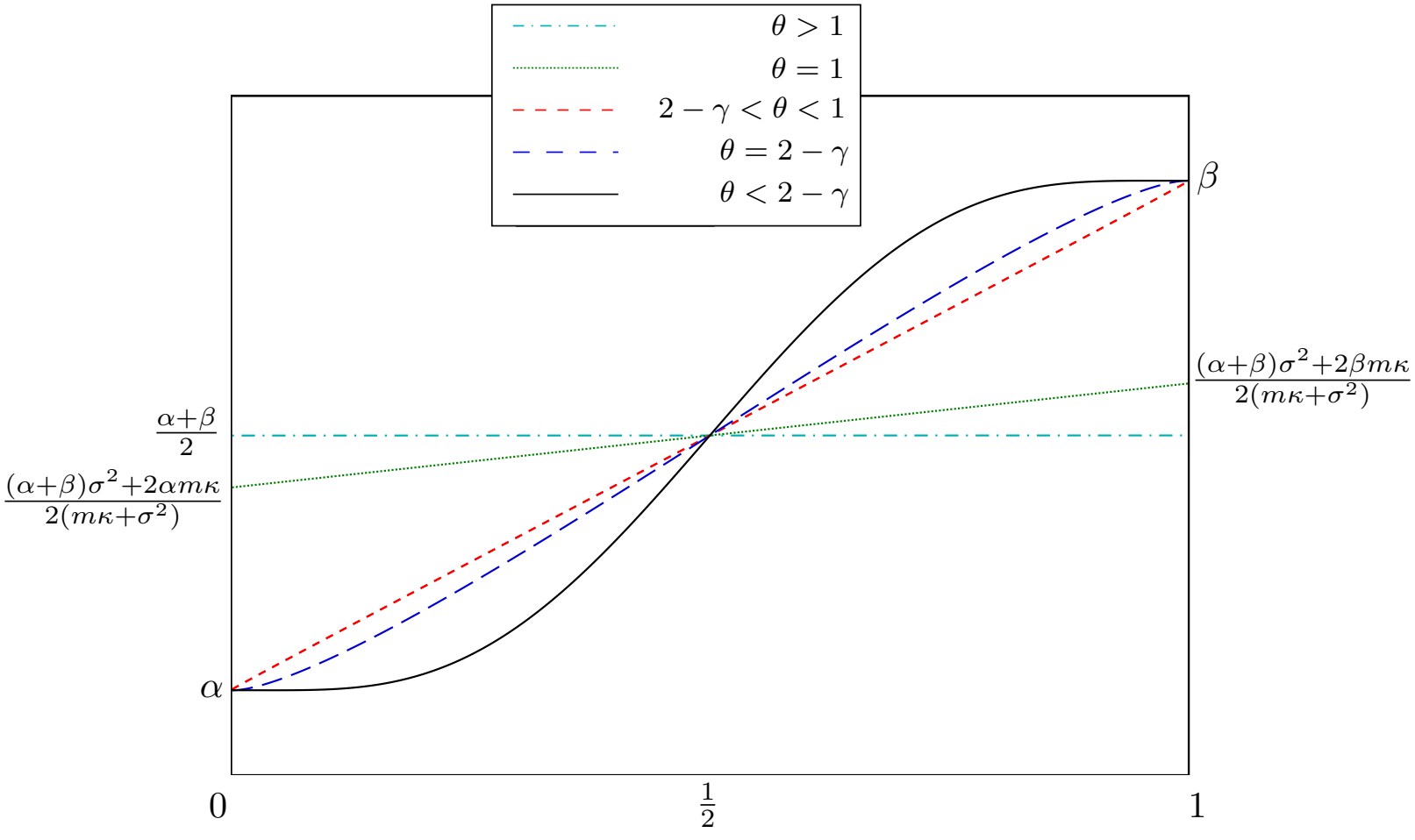
- **Reaction equation:**

$$\partial_t \varrho_t(q) = \kappa(V_0(q) - V_1(q)\varrho_t(q))$$

Above

$$V_1(q) = \frac{c_\gamma}{\gamma}\left(\frac{1}{q^\gamma} + \frac{1}{(1-q)^\gamma}\right)$$
$$V_0(q) = \frac{c_\gamma}{\gamma}\left(\frac{\alpha}{q^\gamma} + \frac{\beta}{(1-q)^\gamma}\right).$$

Stationary solutions:



What about $\gamma \in (1,2)$?

We will get a collection of fractional reaction-diffusion equations

$$\partial_t \varrho_t(q) = \mathbb{L}_\kappa \varrho_t(q) + \kappa V_0(q).$$

where the operator $\mathbb{L}_\kappa = \mathbb{L} - \kappa V_1$, $\mathbb{L}$ is the regional fractional laplacian and

$$V_1(q) = \frac{c_\gamma}{\gamma}\Big(\frac{1}{q^\gamma} + \frac{1}{(1-q)^\gamma}\Big)$$
$$V_0(q) = \frac{c_\gamma}{\gamma}\Big(\frac{\alpha}{q^\gamma} + \frac{\beta}{(1-q)^\gamma}\Big).$$

The operator $\mathbb{L}$:

Let $(-\Delta)^{\gamma/2}$ be the fractional Laplacian of exponent $\gamma/2$ which is defined on the set of functions $H : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{|H(q)|}{(1+|q|)^{1+\gamma}} dq < \infty$$

by (provided the limit exists)

$$-(-\Delta)^{\gamma/2} H(q) = c_\gamma \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \mathbf{1}_{|u-q| \geq \varepsilon} \frac{H(q) - H(u)}{|u-q|^{1+\gamma}} du.$$

Let $\mathbb{L}$ be the regional fractional Laplacian on $[0,1]$, whose action on functions $H \in C_c^\infty(0,1)$ is given by

$$\begin{aligned}(\mathbb{L}H)(q) &= -(-\Delta)^{\gamma/2} H(q) + V_1(q) H(q) \\ &= c_\gamma \lim_{\varepsilon \to 0} \int_0^1 \mathbf{1}_{|u-q| \geq \varepsilon} \frac{H(u) - H(q)}{|u-q|^{1+\gamma}} du, \quad q \in (0,1).\end{aligned}$$

The fractional Sobolev space:

Definition

The Sobolev space $\mathcal{H}^{\gamma/2}$ consists of all square integrable functions $g : (0,1) \to \mathbb{R}$ such that $\|g\|_{\gamma/2} < \infty$, with

$$\|g\|_{\gamma/2} := \langle g, g \rangle_{\gamma/2} = \frac{c_\gamma}{2} \iint_{[0,1]^2} \frac{(g(u) - g(q))^2}{|u-q|^{1+\gamma}} du dq.$$

The space $L^2(0,T;\mathcal{H}^{\gamma/2})$ is the set of measurable functions $f : [0,T] \to \mathcal{H}^{\gamma/2}$ such that $\int_0^T \|f_t\|^2_{\mathcal{H}^{\gamma/2}} dt < \infty$ where $\|f_t\|^2_{\mathcal{H}^{\gamma/2}} := \|f_t\|^2 + \|f_t\|^2_{\gamma/2}$.

Weak solution of $\partial_t \varrho_t(q) = \mathbb{L}_\kappa \varrho_t(q) + \kappa V_0(q)$ with Dir.:

Let $g : [0,1] \to [0,1]$ be measurable. We say
$\varrho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the PDE above if:

- $\varrho \in L^2(0,T;\mathcal{H}^{\gamma/2})$ and
  $\int_0^T \int_0^1 \left\{ \frac{(\alpha - \varrho_t(q))^2}{q^\gamma} + \frac{(\beta - \varrho_t(q))^2}{(1-q)^\gamma} \right\} dq\, dt < \infty,$
- For all $t \in [0,T]$ and any function $H \in C_c^{1,\infty}([0,T] \times (0,1))$:

$$\int_0^1 \varrho_t(q) H_t(q) - g(q) H_0(q)\, dq$$
$$- \int_0^t \int_0^1 \varrho_s(q) \Big(\partial_s + \mathbb{L}_\kappa \Big) H_s(q)\, dq ds$$
$$- \kappa \int_0^t \int_0^1 V_0(q) H_s(q) dq\, ds = 0.$$

Characterizing limit points:

$$N^\gamma \mathcal{L}_N(\langle \pi_s^N, H\rangle) = \frac{N^\gamma}{N} \sum_{x,y\in\Lambda_N} p(y-x)\left[H(\tfrac{y}{N}) - H(\tfrac{x}{N})\right]\eta_s(x)$$

$$+\frac{\kappa N^\gamma}{N}\sum_{x\in\Lambda_N}(Hr_N^-)(\tfrac{x}{N})(\alpha-\eta_s(x)) + \frac{\kappa N^\gamma}{N}\sum_{x\in\Lambda_N}(Hr_N^+)(\tfrac{x}{N})(\beta-\eta_s(x)).$$

For H with compact support in $[a, 1-a]$ for $a \in (0,1)$ we have

$$\lim_{N\to\infty}\left|N^\gamma \sum_{y\in\Lambda_N} p(y-x)\left[H(\tfrac{y}{N}) - H(\tfrac{x}{N})\right] - (\mathbb{L}H)(\tfrac{x}{N})\right| = 0,$$

$$\lim_{N\to\infty}\left|N^\gamma (r_N^-)(\tfrac{x}{N}) - r^-(\tfrac{x}{N})\right| = 0,$$

$$\lim_{N\to\infty}\left|N^\gamma (r_N^+)(\tfrac{x}{N}) - r^+(\tfrac{x}{N})\right| = 0$$

uniformly in $[a, 1-a]$.

Characterizing limit points:

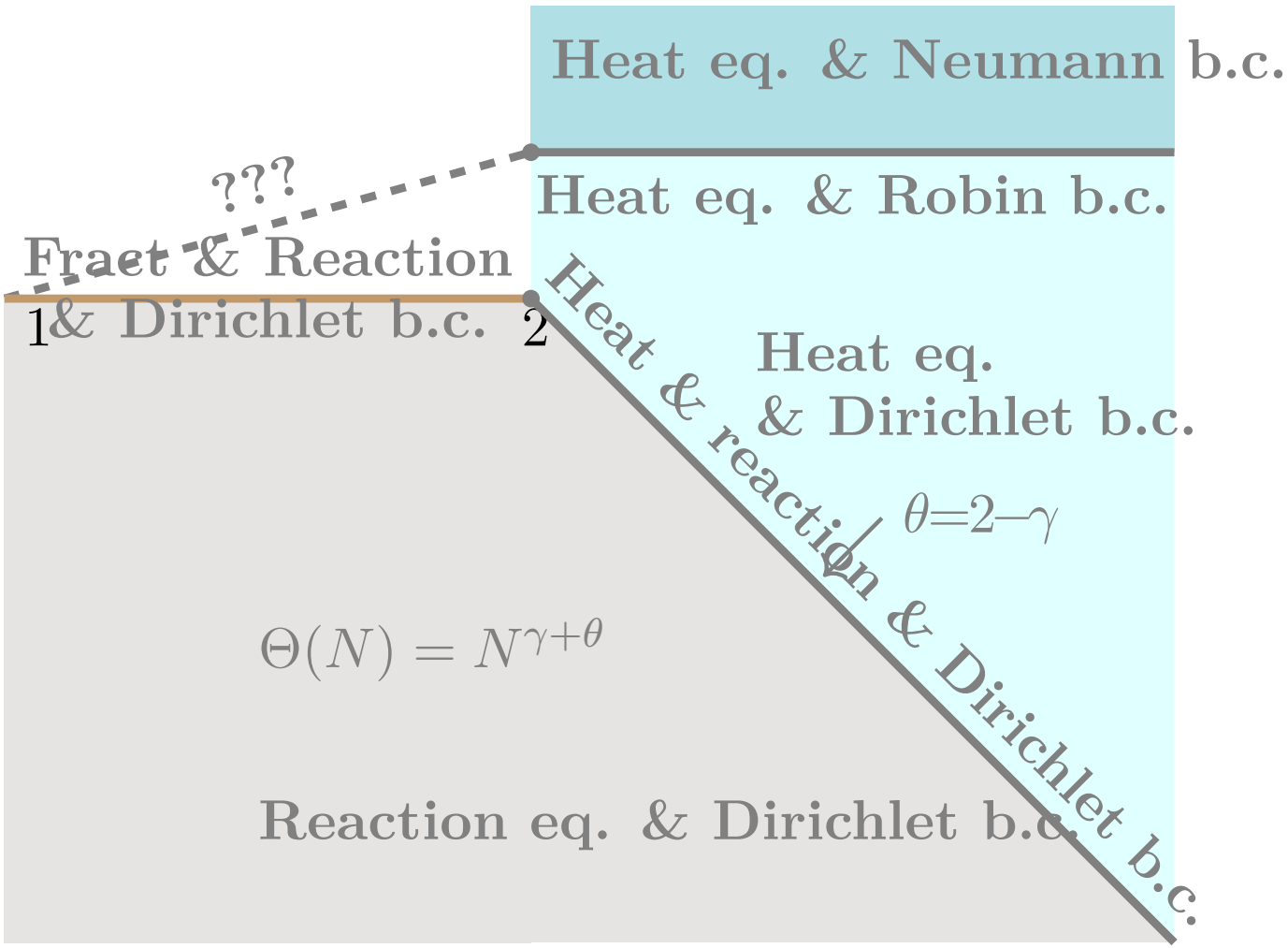
Thus, the first term on the right hand side above can be replaced by

$$\langle \pi_t^N, \mathbb{L}H \rangle \to \int_0^1 (\mathbb{L}H)(q)\varrho_t(q)dq,$$

as N goes to ∞.
The other terms can be replaced by
$\kappa\langle \alpha - \pi_t^N, Hr^- \rangle + \kappa\langle \beta - \pi_t^N, Hr^+ \rangle$ which converges, as N goes to ∞, to

$$\kappa \int_0^1 H(q)r^-(q)(\alpha - \varrho_t(q))dq + \kappa \int_0^1 H(q)r^+(q)(\beta - \varrho_t(q))dq$$
$$= \kappa \int_0^1 H(q)V_0(q)dq - \kappa \int_0^1 H(q)V_1(q)\varrho_t(q)dq.$$

The fast boundary case for $\gamma \in (1,2)$:



Bernardin, C., P.G., Jiménez-Oviedo, B. (2021): *A microscopic model for a one parameter class of fractional laplacians with Dirichlet boundary conditions*, Arch. Ration. Mech. Anal., 239, 1–48.

Which boundary conditions for $\theta > 0$?

For $\theta > 0$ small and $\gamma \in (1, 2)$ the solution should correspond to the solution when $\kappa = 0$. Supported by the result:

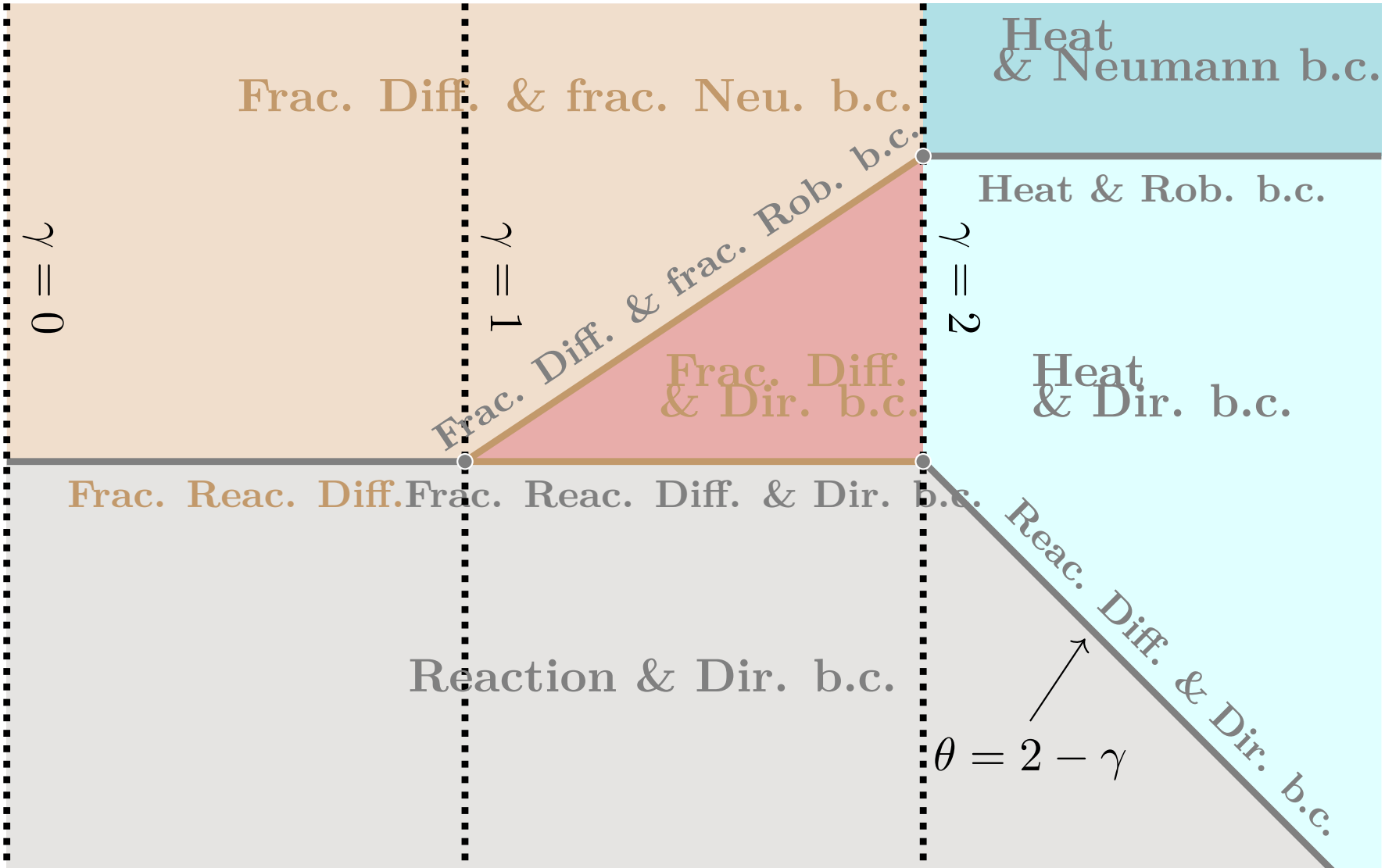


Let $g : [0, 1] \to [0, 1]$ be measurable and ϱ^κ be the weak solution of

$$\partial_t \varrho_t(q) = \mathbb{L}_\kappa \varrho_t(q) + \kappa V_0(q),$$

with Dirichlet boundary conditions and initial condition $g(\cdot)$. Then ϱ^κ converges strongly to ϱ^0 in $L^2(0, T; \mathcal{H}^{\gamma/2})$ as κ goes to 0, where ϱ^0 is the weak solution of the equation with $\kappa = 0$ and initial condition $g(\cdot)$.

The whole picture:



Integration by parts formula:



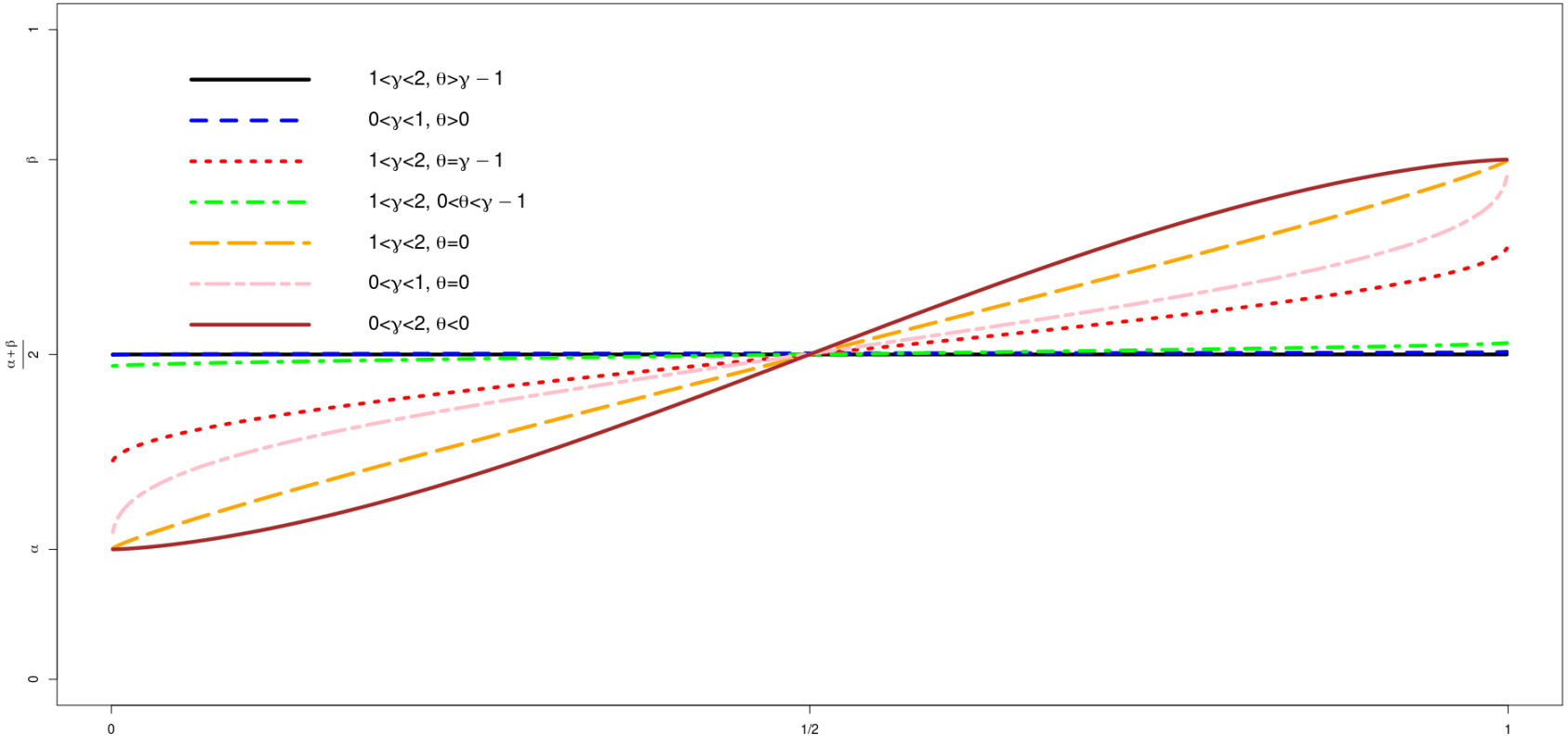
Let $\gamma \in (1,2)$ and $g \in C^2([0,1])$. Let $f : [0,1] \to \mathbb{R}$ be such that $u \to f(u)u^{1-\gamma}$ and $u \to f(u)(1-u)^{1-\gamma}$ are in $C^2([0,1])$. Then,

$$\langle -\mathbb{L}g, f\rangle = \chi_\gamma \left[g(1)D^\gamma f(1) - g(0)D^\gamma f(0)\right] + \langle -\mathbb{L}f, g\rangle$$

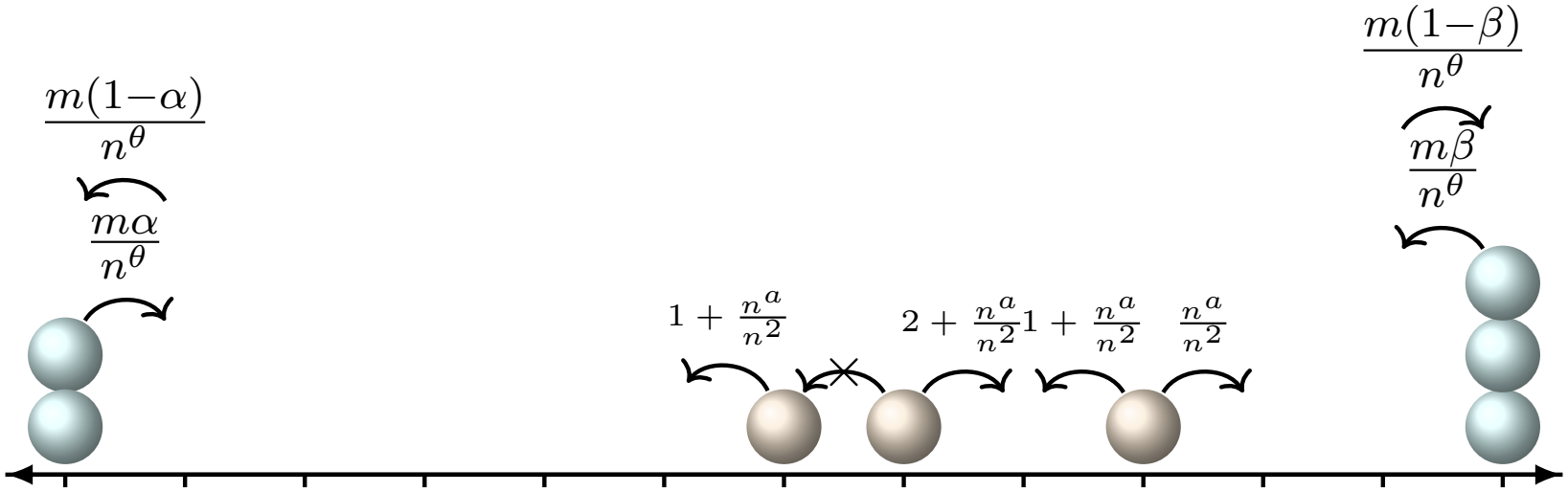
where χ_γ is a constant and $(D^\gamma f)(0) = \lim_{u\to 0^+} f'(u)u^{2-\gamma}$, and $(D^\gamma f)(1) = \lim_{u\to 1^-} f'(u)(1-u)^{2-\gamma}$.

$$\begin{cases} \partial_t \varrho_t^{\hat\kappa}(u) = \mathbb{L}\varrho_t^{\hat\kappa}(u), \quad (t,u) \in [0,T]\times(0,1), \\ \chi_\gamma (D^\gamma \varrho_t^{\hat\kappa})(0) = \hat\kappa m(\alpha - \varrho_t^{\hat\kappa}(0)), \quad \chi_\gamma (D^\gamma \varrho_t^{\hat\kappa})(1) = \hat\kappa m(\beta - \varrho_t^{\hat\kappa}(1)), \\ \varrho_0^{\hat\kappa}(u) = g(u), \quad u \in (0,1), \end{cases}$$

Stationary solutions:



Porous medium model:



$$\begin{cases} \partial_t \rho_t(u) = \Delta\,(\rho_t(u))^2, \quad (t,u) \in (0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, \quad t \in (0,T], \end{cases}$$

$$\begin{cases} \partial_t \rho_t(u) = \Delta\,(\rho_t(u))^2, \quad (t,u) \in (0,T] \times (0,1), \\ \partial_u (\rho_t(0))^2 = \kappa(\rho_t(0) - \alpha), \quad t \in (0,T], \\ \partial_u (\rho_t(1))^2 = \kappa(\beta - \rho_t(1)), \quad t \in (0,T], \end{cases}$$

On $\mathbb{Z}$

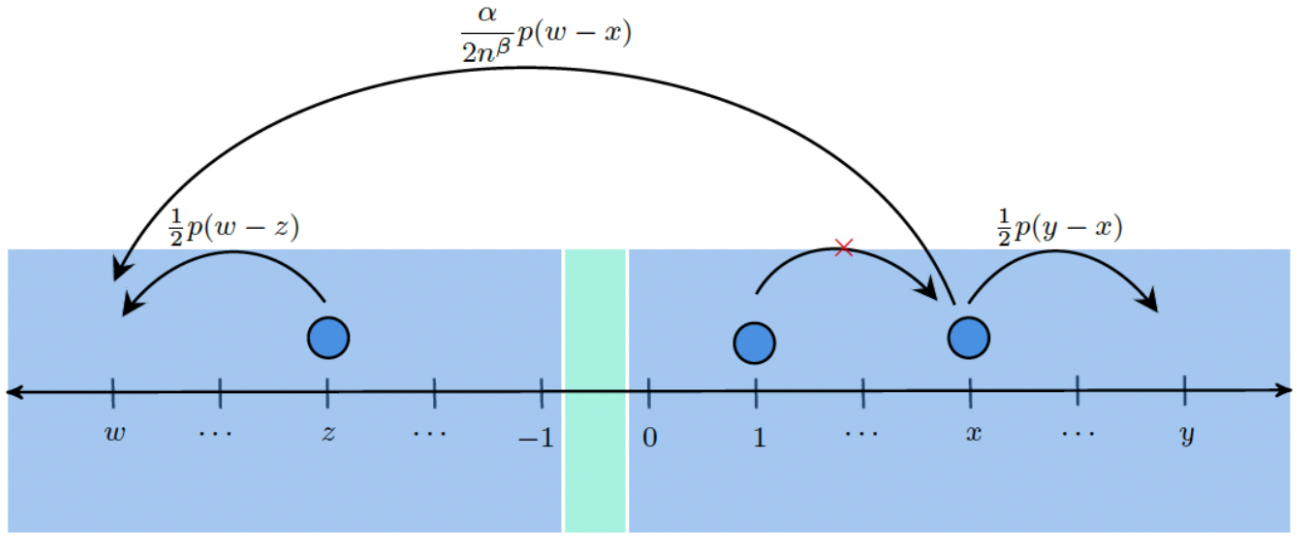
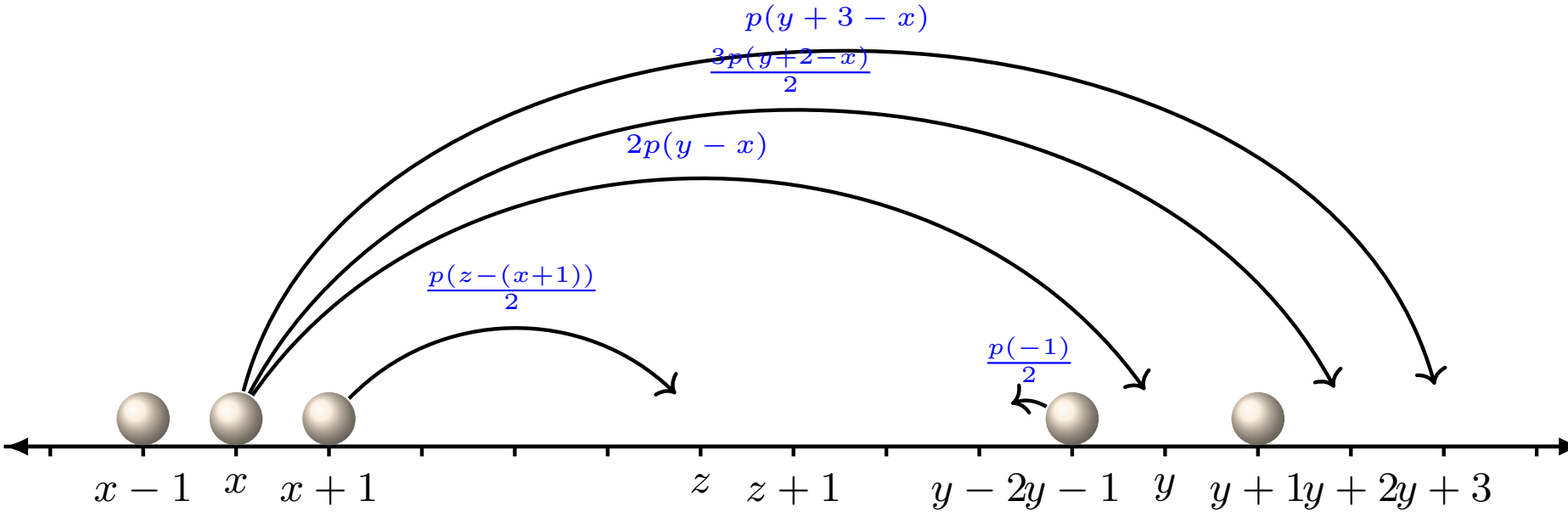


FIGURE 1. Dynamics of the long-jumps symmetric exclusion with slow bonds.

$$\partial_t \varrho(t,u) = \big[-(-\Delta)^{\frac{\gamma}{2}} \varrho\big](t,u), (t,u) \in (0,T] \times \mathbb{R},$$

$$\begin{cases} \partial_t \varrho(t,u) = \big[-(-\Delta)^{\frac{\gamma}{2}}_{\mathbb{R}^*} \varrho\big](t,u), (t,u) \in [0,T] \times \mathbb{R}^*, \\ D^\gamma \varrho(t,0^+) = D^\gamma \varrho(t,0^-) = \frac{\kappa}{C_\gamma}[\varrho(t,0^+) - \varrho(t,0^-)], t \in (0,T], \end{cases}$$

Fractional Porous medium:



$$\partial_t \varrho(t,u) = [-(-\Delta)^{\gamma/2}\varrho^2](t,u), (t,u) \in [0,T] \times \mathbb{R}.$$

More than one conserved quantity:

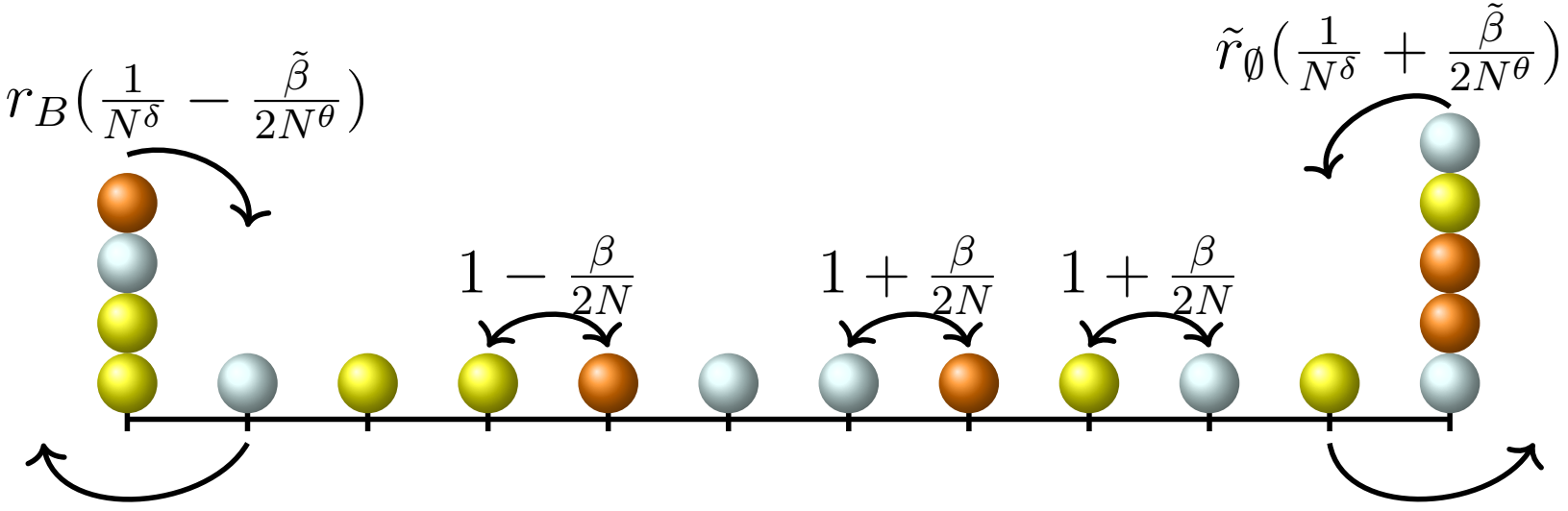


Figure: Particles of species A in yellow, B in orange and C in light cyan.

$$\begin{cases} \partial_t \rho^\alpha = \Delta\, \rho^\alpha + \beta \nabla[\rho^\alpha(\rho^{\alpha+1} - \rho^{\alpha+2})], \\ \rho_t^\alpha(0) = r_\alpha, \quad \rho_t^\alpha(1) = \tilde{r}_\alpha, \quad t \in (0,T], \\ \rho_0^\alpha(u) = \mathfrak{g}^\alpha(u), \quad u \in [0,1], \end{cases} \qquad \alpha \in \{A, B, C\}.$$

For the future:

- Higher dimensions? Other domains!
- What about hydrostatics?
- Fluctuations? Fields evolving in different times scales/frames!
- Other boundary conditions? Non linear boundary conditions?

Thank you very much!!!