Effective results in the three-dimensional minimal model program

Yuri Prokhorov

Steklov Institute

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Minimal model program (MMP): Overview

Birational algebraic geometry studies *algebraic varieties* up to *birational equivalence*.
The aim of the MMP is to find a *good representative* in a fixed birational equivalence class.
Starting with an arbitrary smooth projective variety one can perform a finite number of elementary transformations, called *divisorial contractions* and *flips*, and at the end obtain a variety which is simpler in some sense.

Minimal model program (MMP): Overview

Most parts of the MMP are completed in arbitrary dimension, however, in full generality (including abundance) MMP is proved only in dimensions ≤ 3. But even 3-dimensional MMP is not understood well. One of the basic remaining problems here is the following:

describe all the intermediate steps and the outcome of the MMP.

Since the MMP for surfaces is classical, the first non-trivial case is the 3-dimensional one.
On the other hand, dimension 3 is the last dimension where one can expect effective results: in higher dimensions classifications become very complicated and unreasonably long.

Singularities

It turns out that to proceed with the MMP in dimension ≥ 3 one has to work with varieties admitting certain types of very mild singularities.

Definition

A normal variety is *$\mathbb{Q}$-factorial* if some multiple nD of any Weil divisor D on X is Cartier, i.e. nD is locally given by one equation.

Definition

A normal algebraic variety (or an analytic space) X is said to have *terminal* singularities if some multiple of the canonical Weil divisor K_X is Cartier and for any birational morphism $f : Y \to X$ one can write

$$K_Y = f^* K_X + \sum a_i E_i,$$

where E_i are all the exceptional divisors and $a_i > 0$ for all i.

The smallest positive m such that mK_X is Cartier is called the *Gorenstein index* of X.

If $m = 1$, then X is Gorenstein.

Basic operations in the MMP

- A *contraction* is a proper surjective morphism $f : X \to Z$ of normal varieties with connected fibers.
- The *exceptional locus* of a contraction f is the subset $\mathrm{Exc}(f) \subset X$ of points at which f is not an isomorphism.
- A *Mori contraction* is a contraction $f : X \to Z$ such that
  - the variety X has at worst *terminal $\mathbb{Q}$-factorial* singularities,
  - the anticanonical class $-K_X$ is f-ample, and
  - the relative Picard number $\mathrm{rk}\,\mathrm{Pic}(X/Z)$ equals 1.

A Mori contraction $f : X \to Z$ is said to be

- *divisorial* if it is birational and $\mathrm{codim}\,\mathrm{Exc}(f) = 1$,
- *flipping* if it is birational and $\mathrm{codim}\,\mathrm{Exc}(f) \geq 2$,
- *Mori fiber space (MFS)* if $\dim(Z) < \dim(X)$.

Steps of the MMP

Divisorial contraction: $f : X \to Z$

Then Z has terminal $\mathbb{Q}$-factorial singularities and $\operatorname{rk}\operatorname{Pic}(Z) < \operatorname{rk}\operatorname{Pic}(X)$.
Proceed replacing X with Z.

Flipping contraction: $f : X \to Z$

Then Z is not terminal (and not $\mathbb{Q}$-Gorenstein).
To proceed we need a *flip*:

$$\begin{tikzcd} X \arrow[rr, dashed, "\text{flip}"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

where

- f^+ is a contraction with $\operatorname{codim}\operatorname{Exc}(f^+) \geq 2$,
- X^+ has terminal $\mathbb{Q}$-factorial singularities, and
- K_{X^+} is f^+-ample.

Proceed replacing X with X^+.

Outcome of the MMP

The MMP should terminate with either a minimal model or Mori fiber space. These are two essentially different possibilities for the outcome:

- **Minimal model:** a variety X with terminal singularities is *minimal* if K_X is *numerically effective (nef)*.
- **Variety with Mori fiber space (MFS) structure**
  In dimension 3 for MFS $f : X \to Z$ there are three possibilities.
  - $\dim(Z) = 2$: any component of a fiber is a smooth rational curve, f is called *$\mathbb{Q}$-conic bundle*;
  - $\dim(Z) = 1$: the general fiber is a smooth del Pezzo surface, f is called *$\mathbb{Q}$-del Pezzo fibration*;
  - $\dim(Z) = 0$: then X is a Fano variety with at worst terminal $\mathbb{Q}$-factorial singularities and $\mathrm{Pic}(X) \simeq \mathbb{Z}$.
    We call such varieties *$\mathbb{Q}$-Fano*.

Three-dimensional MMP was completed in full generality in works of Reid, Mori, Kawamata, Kollár, Shokurov, Miyaoka and others.

Consequences

Theorem (Miyaoka-Mori 86, Miyaoka 88)

Let X *be a smooth algebraic variety of dimension* ≤ 3*. Then the following are equivalent:*

- $P_n(X) := \dim H^0(X, nK_X) = 0$ *for all* $n > 0$*;*
- X *is uniruled, i.e. it is covered by rational curves;*
- *there is a birational transformation* $X \dashrightarrow X'$*, where* X' *has MFS structure* $f : X' \to Z$*.*

Theorem (Kollár-Miyaoka-Mori 92, Mumford's conjecture)

Let X *be a smooth algebraic variety of dimension* ≤ 3*. Then the following are equivalent:*

- $H^0(X, (\Omega^1_X)^{\otimes n}) = 0$ *for all* $n > 0$*;*
- X *is rationally connected, i.e. two general points of* X *can be connected by a rational curve;*
- *there is a birational transformation* $X \dashrightarrow X'$*, where* X' *has MFS structure* $f : X' \to Z$ *with rationally connected* Z*.*

Terminal 3-fold singularities

Terminal 3-fold singularities are isolated.

Theorem (M. Reid)

Let $(X \ni P)$ be an analytic germ of a 3-dimensional terminal singularity of (Gorenstein) index $m \geq 1$.
Then $(X \ni P)$ is the quotient

$$(X \ni P) = \left(X^\sharp \ni P^\sharp\right)/\boldsymbol{\mu}_m$$

where $\left(X^\sharp \ni P^\sharp\right)$ is a terminal hypersurface singularity and $\boldsymbol{\mu}_m$ is a cyclic group of order m that acts freely outside $P^\sharp$.

The corresponding cover

$$\pi : \left(X^\sharp \ni P^\sharp\right) \longrightarrow (X \ni P)$$

is called the *index-one cover*.
All possibilities for $\left(X^\sharp \ni P^\sharp\right)$ and the action of $\boldsymbol{\mu}_m$ up to analytic isomorphisms are classified [Mori 85].

Example of a terminal 3-fold singularity

Type $\mathrm{cA/m}$ (main series)

Consider an isolated singularity

$$xy + \phi(z^m, t) = 0$$

and the diagonal action of $\boldsymbol{\mu}_m(1, -1, a, 0)$, i.e.

$$(x, y, z, t) \longmapsto \left(\zeta x,\ \zeta^{-1} y,\ \zeta^a z,\ t\right)$$

where $\zeta^m = 1, \quad \gcd(m,\ a) = 1$. Then the quotient

$$\{xy + \phi(z^m, t) = 0\} \,/ \boldsymbol{\mu}_m(1, -1, a, 0)$$

is a terminal singularity.

The extreme case $\phi = t$ is not excluded.
Then $(X \ni P) \simeq \mathbb{C}^3/\boldsymbol{\mu}_m(1, -1, a)$ is called terminal *cyclic quotient singularity*.

General elephant

Conjecture (Reid's general elephant principle)

Let $f : X \to (Z \ni o)$ be a 3-fold Mori contraction, where $(Z \ni o)$ is a small neighborhood.
Then a general member $D \in |-K_X|$ is a normal surface with Du Val (ADE) singularities.

Example (Shokurov, Reid)

Let X be a Fano 3-fold with *Gorenstein* terminal (and even canonical) singularities.
Then a general member $D \in |-K_X|$ is a K3 surface with only Du Val singularities.

Warning

Reid's general elephant fails for $\mathbb{Q}$-Fano 3-folds.
However there are only a few counterexamples.

General elephant (local version)

Example (Reid)

Let $(X \ni P)$ be a 3-fold terminal singularity of index m. Then a general member $D \in |-K_X|$ is a Du Val singularity.
Furthermore, assume that $m > 1$. Let $\pi : X' \to X$ be the index-1 cover and let $D' := \pi^{-1}(D)$. Then $D' \longrightarrow D$ is one of the following:

- cA/m $\quad A_{k-1} \xrightarrow{m:1} A_{km-1}$
- cAx/4 $\quad A_{2k-2} \xrightarrow{4:1} D_{2k+1}$
- cD/3 $\quad D_4 \xrightarrow{3:1} E_6$
- cAx/2 $\quad A_{2k-1} \xrightarrow{2:1} D_{k+2}$
- cD/2 $\quad D_{k+1} \xrightarrow{2:1} D_{2k}$
- cE/2 $\quad E_6 \xrightarrow{2:1} E_7$

This division corresponds to local analytic classification of terminal 3-fold singularities.

Extremal curve germs

Definition

An *complex analytic* germ $(X \supset C)$ of a 3-fold X with terminal singularities along a reduced connected complete curve C is called an *extremal curve germ* if there exists a contraction

$$f : (X \supset C) \longrightarrow (Z \ni o)$$

such that

- $C = f^{-1}(o)_{\mathrm{red}}$ and
- $-K_X$ is f-ample.

An extremal curve germ is said to be *irreducible* if so its central fiber C is.

A *general hyperplane section* is the divisor $H := f^{-1}(H_Z) \in |\mathscr{O}_X|_C$, where $H_Z \subset Z$ is given by $t = 0$ for general $t \in \mathfrak{m}_{o,Z}$.

Extremal curve germs

If $f : X \to Z$ is a 3-fold Mori contraction with fibers of dimension ≤ 1 and C is a 1-dimensional component of a fiber, then $(X \supset C)$ is an extremal curve germ.

Irreducible extremal curve germs and divisorial contractions to points are building blocks of the 3-dimensional MMP.

Three types of extremal curve germs:

- *flipping* if is f birational and does not contract divisors;
- *divisorial* if the exceptional locus is two-dimensional;
- *$\mathbb{Q}$-conic bundle germ* if Z is a surface.

Assumption:

We assume that X is not Gorenstein.

Extremal curve germs: examples

Example (Flip of index 2, Francia's flip)

Take a non-$\mathbb{Q}$-Gorenstein singularity

$$Z = \{x_1x_3 + x_2x_4 = 0\}/\boldsymbol{\mu}_2(1,1,0,0)$$

and consider two blowups:

$$\begin{tikzcd}[column sep=small]
(X \supset C) \arrow[rr, dashed, "\text{flip}"] \arrow[dr, bend right, "{\substack{\text{blowup of} \\ \{x_2 = x_3 = 0\} \\ /\boldsymbol{\mu}_2}}"'] & & (X^+ \supset C^+) \arrow[dl, bend left, "{\substack{\text{blowup of} \\ \{x_1 = x_2 = 0\} \\ /\boldsymbol{\mu}_2}}"] \\
& (Z \ni o) &
\end{tikzcd}$$

Here X has a unique cyclic quotient singularity of index 2 and X^+ is smooth.

Classification of extremal curve germs: general strategy

Step 1. Local description of possible singularities $(P \in X) \supset C$

$\Downarrow$

Step 2. General elephant $D \in |-K_X|$

$\Downarrow$

Step 3. General hyperplane section $H \in |\mathscr{O}_X|_C$

$\Downarrow$

Step 4. Reconstruction of $f : (X \supset C) \to Z$ from H

General elephant: irreducible central fiber

Theorem (Mori, Kollár-Mori, Mori-Prokhorov)

Let $(X \supset C)$ be an irreducible extremal curve germ. Then a general member $D \in |-K_X|$ has only Du Val singularities.

Idea of the proof

Essentially, there are three methods to find a good elephant $D \in |-K_X|$.

- Construct a general member $D \in |-K_X|$ locally near a singular point $P \in C \subset X$ and show that D is the global member of $|-K_X|$ along C (works in the cases $D \not\supset C$).
- Extend a good element $D \in |-K_X|$ from a general surface $S \in |-2K_X|$ (bi-elephant, which is supposed to be "good enough").
- The good element $D \in |-K_X|$ is recovered as the formal Weil divisor $\varprojlim C_n$ of the completion $X^\wedge$ of X along C, where C_n are subschemes with support C constructed by using certain inductive procedure [Mori88], [Mori,– 08].

General elephant: consequences and generalizations

- The existence of general elephant $D \in |-K_X|$ for flipping extremal curve germs implies the existence of 3-fold flips [Kawamata 88].
- If $f : X \to Z$ is a 3-fold conic bundle contraction, then the base Z has at worst *Du Val* singularities of type A_n.

From general elephant to hyperplane sections

Example-Construction

Let $f : (X \supset C) \to Z$ be an extremal curve germ. Assume that a general member $D \in |-K_X|$ is Du Val. Consider the Stein factorization:

$$f_D : D \xrightarrow{f'} D' \longrightarrow f(D) \qquad\qquad D' = f(D) \text{ if } f \text{ is birational}$$

Then D' is Du Val at $f'(C)$.

(*) Assume that $D' \ni f'(C)$ is of type A_n.

Then by the *Inversion of Adjunction* for a general $H \in |\mathscr{O}_X|_C$ the pair $(X, H + D)$ is log canonical (lc).
If moreover $D \supset C$, then H is normal and has only cyclic quotient singularities.

Definition

If $(X \supset C)$ satisfies (*), then $(X \supset C)$ is said to be *semistable*.
Otherwise, $(X \supset C)$ is called *exceptional*.

Reconstruction germs from general $H \in |\mathscr{O}_X|_C$

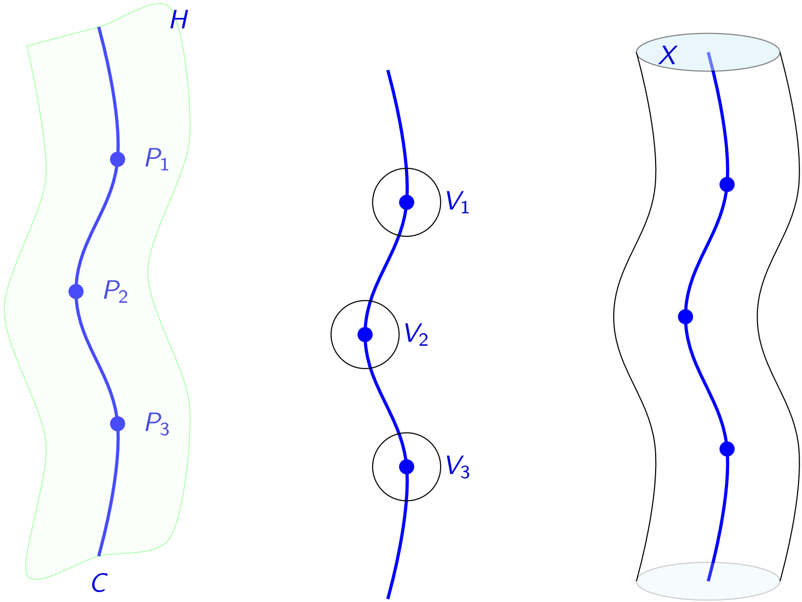
Initial data.

- A normal surface germ $(H \supset C)$ and a K_H-negative contraction $f_H : H \to H_Z$ such that C is a fiber.
- Near each singular point $P_i \in H$ there is a small one-parameter deformation H_i^t, $t \in \mathbb{D} \subset \mathbb{C}$ of a neighborhood $U_i \subset H$ of P_i such that the total space $V_i = \bigcup H_i^t$ has a terminal singularity at P_i.

Since $R^2 f_* \mathcal{T}_H = 0$, where $\mathcal{T}_H = (\Omega_H^1)^\vee$ is the tangent sheaf, the deformations are unobstructed. Hence there is a global one-parameter deformation H^t of H inducing a local deformation of H_i^t near P_i.

Then we have a 3-fold $X = \bigcup H^t \supset C$ with $H \in |\mathscr{O}_X|_C$ such that locally near P_i it has the desired structure. We can extend f_H to a contraction

$$f : X \longrightarrow Z, \qquad f(C) = \text{point}$$



H
P1
P2
P3
C
V1
V2
V3
X

Semistable extremal curve germs and T-singularities

Definition (Kollár, Shepherd-Barron 1988)

A 2-dimensional *quotient* singularity $F \ni P$ is called *T-singularity* if it admits a smoothing in a *$\mathbb{Q}$-Gorenstein* family $X \to \mathbb{D} \subset \mathbb{C}$.

In this case the total family X is terminal.

Classification of T-singularities :

- Du Val (ADE) points;
- cyclic quotients of types

$$\mathbb{C}^2/\boldsymbol{\mu}_{dn^2}(1,\, dna-1),$$

  where $\gcd(n,\, a) = 1$.

Observation

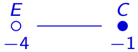
Let $(X \supset C)$ be a semistable extremal curve germ. Assume that $H \in |\mathscr{O}_X|_C$ is normal. Then the singularities of H are of type T.

Example of semistable extremal curve germs

The singularity

$$\mathbb{C}^2/\boldsymbol{\mu}_4(1,1)$$

is of type T. The exceptional locus of its minimal resolution consists of one smooth rational curve E with $E^2 = -4$.
Consider a smooth surface $\tilde{H}$ containing a configuration of smooth rational curves with the following dual graph:



Contracting E we obtain a surface germ $(H \supset C)$ with a T-singularity as above. An 1-parameter deformation produces Francia's flip.

Classification of **irreducible** curve germs

Local [Mori88], [Mori-Prokhorov08]

The configuration of singular points and the local analytic description of $(X \ni P) \supset C$ is given.

Global [Kollár-Mori92], [Mori02], [Mori-Prokhorov08-21]

The general hyperplane sections $H \in |\mathscr{O}_X|_C$ and their 1-parameter deformations are described in all cases except for a few cases: non-flipping cases of (kAD) and $(\mathrm{k3A})$.

Classification of **irreducible** curve germs

[Mori88], [Kollár-Mori92], [Mori-Prokhorov08-21]

- Semistable

(k1A) one non-Gorenstein point of type cA/m
(k2A) two non-Gorenstein points of types cA/m, cA/m'

- Exceptional

$\cdots$ one non-Gorenstein point of type $\mathrm{cD}/2$, $\mathrm{cAx}/2$, or $\mathrm{cE}/2$
$\mathrm{cD}/3$ one non-Gorenstein point which is of type $\mathrm{cD}/3$
(IIA)
(II$^\vee$)
(IIB)
$\Bigg\}$ one non-Gorenstein point of exceptional type $\mathrm{cAx}/4$
(IE$^\vee$) one singular point which is a cyclic quotient of index 8 ($\mathbb{Q}$-conic bundle only)
(IC) one singular point which is a cyclic quotient of index $m \geq 5$ (odd)
(kAD) two singular points of indices $m \geq 3$ (odd) and 2
(k3A) three singular points of indices $m \geq 3$ (odd), 2 and 1

Divisorial contractions to a point

There is an almost complete classification: Y. Kawamata, A. Corti, M. Kawakita, T. Hayakawa, ...

Theorem (M. Kawakita)

Let $f : X \to (Z \ni o)$ be a divisorial Mori contraction that contracts a divisor to a point.
Then a general member $D \in |-K_X|$ is Du Val.

Theorem (M. Kawakita)

Let $f : X \to Z$ be a divisorial Mori contraction that contracts a divisor to a smooth point.
Then f is the weighted blowup with weights $(1, a, b)$, $\quad \gcd(a, b) = 1$.

$\mathbb{Q}$-del Pezzo fibrations

Let $f : X \to Z \ni o$ be a germ of $\mathbb{Q}$-del Pezzo fibration.

- The existence of a good element $D \in |-K_X|$ is not known.
- If $D \in |-K_X|$ is Du Val, then $D \to Z$ is a minimal elliptic fibration obtained by contracting of some (-2)-curves on a smooth one.

Two extreme cases are studied relatively well.

Multiple fibers.

Theorem (Mori – Prokhorov)

Assume that $f^*(o) = m_o F_o$ is a fiber of multiplicity $m_o \geq 2$.
Then $m_o \leq 6$.
Moreover, all the cases $m_o \leq 6$ occur.
The possibilities for the local behavior of F_o near singular points and the possible values of the degree of the generic fiber are described.

$\mathbb{Q}$-del Pezzo fibrations

Fibers with quotient singularities.

Assumption

Let $f : X \to Z \ni o$ be a germ of $\mathbb{Q}$-del Pezzo fibration whose central fiber $F := f^{-1}(o)$ is reduced normal and has only *quotient singularities*.

Theorem (Hacking – Prokhorov)

Let F be as above. Assume additionally that $\mathrm{Pic}(F) \simeq \mathbb{Z}$. Then F belongs to one of the following classes:

- 14 *infinite families of toric surfaces determined by Markov-type equations;*
- *partial smoothing of a toric surface as above;*
- 18 *sporadic families of surfaces of index ≤ 2.*

The variety X is obtained as an 1-parameter deformation space of F.

$\mathbb{Q}$-Fano 3-folds

Theorem (Y. Kawamata ($\dim = 3$), C. Birkar ($\forall \dim$))

The set of all $\mathbb{Q}$-Fano varieties of fixed dimension is bounded.

Remark

- *Smooth* $\mathbb{Q}$-Fano 3-folds are classified by V. Iskovskikh (17 families).
- $\mathbb{Q}$-Fano 3-folds with *Gorenstein* singularities are degenerations of smooth ones (Y. Namikawa).

Basic invariants

- Collection of singularities
- Degree: $(-K_X)^3$ (rational number)
- Fano index: $q(X) := \max\{t \in \mathbb{N} \mid -K_X \underset{\mathbb{Q}}{\sim} tA,\ A \text{ is a Weil divisor}\}$

Effective boundedness of $\mathbb{Q}$-Fano 3-folds

Theorem (Kawamata)

Let X be a $\mathbb{Q}$-Fano 3-fold.

- $\sum \left(m_i - \frac{1}{m_i} \right) < 24$, *where m_i's are the indices of singular points*
- $(-K_X)^3 \leq b$, *where b is an effectively computable (but large) constant*

Theorem (P.)

Let X be a singular $\mathbb{Q}$-Fano 3-fold. Then $(-K_X)^3 \leq 125/2$.
If $(-K_X)^3 = 125/2$, then $X \simeq \mathbb{P}(1,1,1,2)$ (Veronese cone).

Theorem (J. A. Chen & M. Chen)

Let X be a $\mathbb{Q}$-Fano 3-fold. Then $(-K_X)^3 \geq 1/330$.

Theorem (Chen Jiang)

If $(-K_X)^3 = 1/330$ and X is not rational, then
$X \simeq X_{66} \subset \mathbb{P}(1,5,6,22,33)$.

Thank you for attention!