Algebraic geometry in mixed characteristic

Bhargav Bhatt

Institute for Advanced Study
Princeton University
University of Michigan

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Motivating goal

Fix a prime number p.

Theorem (de Rham 1931, Hodge 1941)

Let X be a compact complex Kähler manifold (e.g., $X \subset \mathbf{CP}^m$). Integration of forms over cycles yields

$$H^n(X;\mathbf{C}) \simeq H^n_{dR}(X;\mathbf{C}) \simeq \bigoplus_{i+j=n} H^{i,j}(X).$$

This can be used in both directions:

- *Geometry to topology:* n odd $\rightsquigarrow$ $\dim H^n(X;\mathbf{C})$ is even as $H^{i,j} \simeq \overline{H^{j,i}}$.
- *Topology to geometry:* $\pi_1(X) = 0 \rightsquigarrow$ holomorphic 1-forms on X vanish as $H^{1,0}(X) \subset H^1(X;\mathbf{C}) = 0$.

Defect: Misses torsion in $H^*(X;\mathbf{Z})$ completely!

Goal

Understand $H^*(X;\mathbf{Z}/p^n)$ geometrically (e.g., via forms), at least for X algebraic.

The geometric backdrop: Hensel's p-adic world (1897)

- $\mathbf{Z}_p :=$ ring of p-adic integers
  $= \{\sum_{i \geqslant 0} a_i p^i \mid a_i \in \{0, \ldots, p-1\}\}$

- $\mathbf{F}_p = \mathbf{Z}_p/(p) \xleftarrow{\text{kill } p} \mathbf{Z}_p \xrightarrow{\text{invert } p} \mathbf{Z}_p[\frac{1}{p}] = \mathbf{Q}_p \subset \widehat{\overline{\mathbf{Q}}_p} =: \mathbf{C}_p \overset{\text{non-canonical}}{\cong} \mathbf{C}$
  "$\mathbf{Z}_p$ yields a bridge from characteristic 0 to characteristic p"

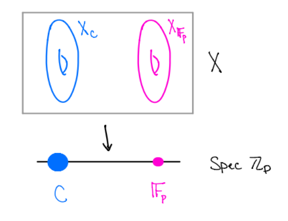
- $X/\mathbf{Z}_p$ smooth projective variety
  $\rightsquigarrow X_{\mathbf{C}_p} =$ smooth projective variety over $\mathbf{C}_p$
  $\rightsquigarrow X_{\mathbf{F}_p} =$ " $\mathbf{F}_p$

Example

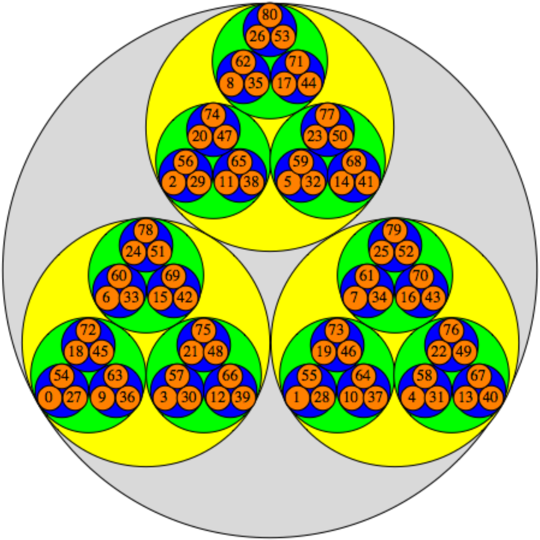
$X =$ Fermat cubic curve

$= V(x^3 + y^3 + z^3) \subset \mathbf{P}^2_{\mathbf{Z}_p}$

($p \neq 3$ to ensure smoothness)



An illustration of $\mathbf{Z}_3$ (by Heiko Knospe)



Source: `http://www.nt.th-koeln.de/fachgebiete/mathe/knospe/p-adic/`

Progress from the 20th century

Fix $X/\mathbf{Z}_p$ smooth projective as before $\rightsquigarrow X_{\mathbf{C}_p}, X_{\mathbf{F}_p}$

Principle (Tate 1966, Grothendieck 1970, Fontaine 1978, ...)

Mod p^n topology of $X_{\mathbf{C}_p}$ $\approx$ Algebraic geometry of $X_{\mathbf{F}_p}$

Last century: principle works well, at least up to bounded p-torsion:

Example (Faltings 1989, Tsuji 1999)

$$H^i(X_{\mathbf{C}_p}; \mathbf{Z}_p) \overset{\text{up to isogeny}}{=} \text{Complicated functor}\left(X_{\mathbf{F}_p}, \mathcal{F}_X\right),$$

where $\mathcal{F}_X$ is some linear algebra data attached to the lift X of $X_{\mathbf{F}_p}$.

Applications include:

- Langlands program (e.g: structure for p-adic Galois representations).
- Birational geometry (e.g: birational invariance of $h^{i,j}$(Calabi Yau))

A 21st century upgrade: prismatic cohomology

Theorem (B-Morrow-Scholze 2016, 2018)

There exists a cohomology theory $H^*_{\Delta}(X; \mathbf{F}_p) \in \mathrm{Mod}^{fg}(\mathbf{F}_p[\![T]\!])$ *such that:*

- *Topological comparison:* $H^*_{\Delta}(X; \mathbf{F}_p)[\frac{1}{T}] \simeq H^*(X_{\mathbf{C}_p}; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathbf{F}_p((T))$
- *Differential comparison:* $H^*_{\Delta}(X; \mathbf{F}_p)/T \approx H^*_{dR}(X_{\mathbf{F}_p})$

- *Partial inspiration*: Breuil and Kisin's work on Galois representations.
- Theorem + Linear Algebra ⇒ $\dim_{\mathbf{F}_p} H^n(X_{\mathbf{C}_p}; \mathbf{F}_p) \leqslant \dim_{\mathbf{F}_p} H^n_{dR}(X_{\mathbf{F}_p})$
- ∃ an integral variant $H^*_{\Delta}(X; \mathbf{Z}_p) \in \mathrm{Mod}^{fg}(\mathbf{Z}_p[\![T]\!])$
- ∃ semistable variant (Česnavičius-Koshikawa 2018, Koshikawa 2020)
- Several distinct constructions by now:
    - Analytic: via perfectoid spaces and the almost purity theorem (BMS1)
    - Homotopical: via K-theory and THH (BMS2)
    - Algebraic: via prisms (B-Scholze)
    - Algebro-geometric: via derived algebraic geometry (Drinfeld, B-Lurie)

The shape of prismatic cohomology

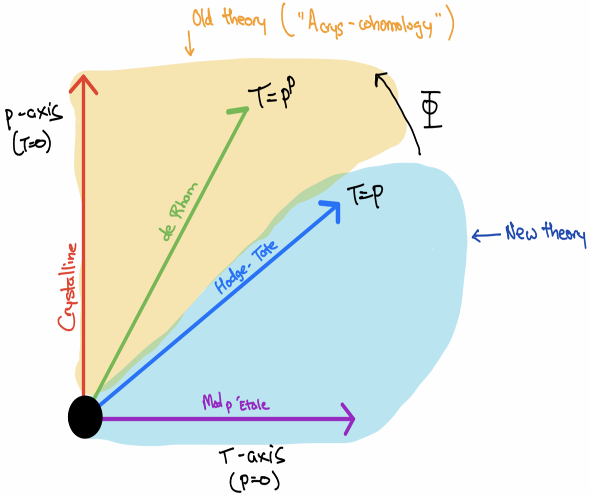


Figure: The fibers of $H^*_{\Delta}(X;\mathbf{Z}_p)$ over $\mathrm{Spec}(\mathbf{Z}_p[\![T]\!])$

Semistable reduction

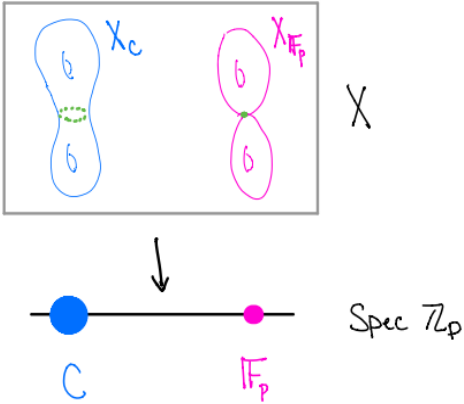


Figure: A picture of genus 2 curve $X_{\mathbf{C}_p}$ with semistable reduction $X_{\mathbf{F}_p}$

Connections to algebraic K-theory

X a (nice) space $\rightsquigarrow K(X) = K$-theory of complex vector bundles on X

Theorem (Atiyah-Hirzebruch 1961, Bott 1957)

For a nice space X, there is a natural "skeletal" filtration on $K(X)$ with $\mathrm{gr}^i \simeq H^*(X; \mathbf{Z}(i))[2i]$.

Quillen 1973: X algebraic variety $\rightsquigarrow K(X) =$ algebraic K-theory of X

Question (Beilinson 1982)

What is the analog of the Atiyah-Hirzebruch(-Bott) theorem for the algebraic K-theory of a ring R or a variety X?

The rational/ℓ-adic case was mostly answered in the $20th$ century (Thomason, Suslin, Gabber, Bloch-Lichtenbaum, Geisser, Levine, Voevodsky, ...).

Theorem (Clausen-Mathew-Morrow 2018, B-Morrow-Scholze 2018)

For a p-complete commutative ring R, there is a natural "motivic" filtration on $K_{et}(R; \mathbf{Z}_p)$ with $\mathrm{gr}^i \simeq H^*_{\mathrm{syn}}(R, \mathbf{Z}_p(i))[2i] \leftsquigarrow$ "syntomic cohomology" (a new object determined by prismatic cohomology).

New calculations in algebraic K-theory

Algebraic construction of prismatic cohomology + previous theorem $\Rightarrow$ a new tool to calculate p-adic K-theory.

Example (Fix $n \geqslant 1$)

- Odd vanishing (B-Scholze 2019): The object $\pi_{\text{odd}} K(R; \mathbf{Z}_p)$ vanishes for many "very large" rings, e.g.,

$$\pi_{odd} K(\mathcal{O}_{\mathbf{C}_p}/p^n) = 0.$$

Prismatic input: q-de Rham complexes, André's flatness lemma

- Even vanishing (Antieau-Krause-Nikolaus 2022): We have

$$\pi_{2k} K(\mathbf{Z}/p^n) = 0 \;\; \forall k \gg 0.$$

Prismatic input: Absolute prismatic cohomology

Both results were previously out of reach for $n > 1$.

Kodaira vanishing in mixed characteristic

Theorem (Kodaira 1953)

Given $X \subset \mathbf{P}^n_{\mathbf{C}}$ smooth projective of dimension d, we have $H^{<d}(X, \mathcal{O}(-1)) = 0$.

Fact: This vanishing fails outside characteristic 0 (Raynaud 1978, Totaro 2021).

Theorem (KV up to finite covers, B 2020)

For $X \subset \mathbf{P}^n_{\mathbf{Z}_p}$ projective of relative dimension d, $\exists$ a finite cover $\pi : Y \to X$ such that

$$\mathrm{Image}\left(H^{<d}(X, \mathcal{O}(-1))_{tors} \xrightarrow{\pi^*} H^{<d}(Y, \pi^*\mathcal{O}(-1))_{tors} \right) = 0.$$

- Formulation inspired by work of Hochster-Huneke and Smith from 1990s. In fact, analog over $\mathbf{F}_p$ follows from their work.
- *Key ingredients:*
  - Prismatic cohomology.
  - Riemann-Hilbert constructions for perverse $\mathbf{F}_p$-sheaves on $X_{\mathbf{C}_p}$ (B-Lurie).

Kodaira vanishing: variants and applications

Theorem (Local KV up to finite covers, B 2020)

Fix a finite extension $\mathbf{Z}_p[x_1, ..., x_n] \subset R$. Then $\exists$ a finite extension $R \subset S$ such that: any relation $\sum_i a_i x_i = 0$ in R/p becomes "trivial" in S/p.

Conceptual reformulation:

Set $R^+ :=$ integral closure R in $\overline{\mathrm{Frac}(R)}$. Then R^+/p is Cohen-Macaulay over R/p.

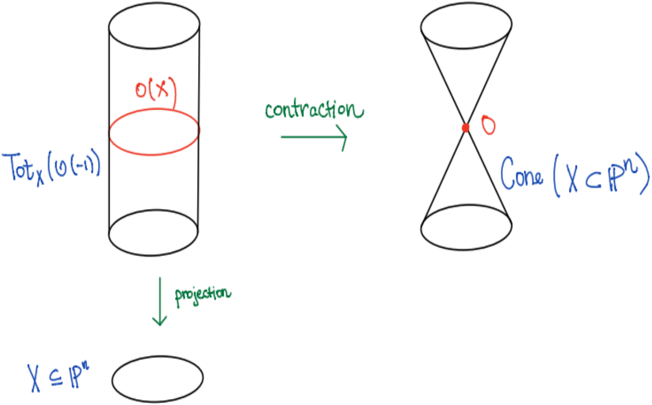
This local variant

- was previously "almost" known for $n \leqslant 2$ (Heitmann 2002).
- has been expected (but not conjectured...) for over 30 years.

Theorem (B-Ma-Patakfalvi-Schwede-Tucker-Waldron-Witaszek 2020, Takamatsu-Yoshikawa 2020)

One can run the minimal model program for arithmetic 3-folds with residue characteristic > 5.

Interlude: Serre's global-to-local dictionary



Projective geometry of $X \subset \mathbf{P}^n$ $\approx$ Local geometry of $\mathrm{Cone}(X \subset \mathbf{P}^n)$ at 0

Some other results featuring prismatic cohomology

- (Madapusi Pera 2016, Ito-Ito-Koshikawa 2018): The Tate conjecture for K3 surfaces in characteristic 2.
- (Colmez-Dospinescu-Nizioł 2019): $H^*(\Omega; \mathbf{Z}_p)$ is p-torsionfree, where $\Omega \subset \mathbf{P}^d_K$ is Drinfeld's p-adic upper half space over a p-adic field K.
- (Farb-Kisin-Wolfson 2021): For $A/\mathbf{C}$ abelian variety and $p \gg 0$, the multiplication by p map $[p] : A \to A$ has essential dimension $\dim(A)$.
- (Zavyalov 2021): Poincaré duality for $\mathbf{Z}_p$-étale cohomology in p-adic analytic geometry (also Gabber 2015, Mann 2022).

Thank you for listening!

Image a prism by Brett Jordan via Flickr