Recent progress towards Hadwiger's conjecture

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Combinatorics section

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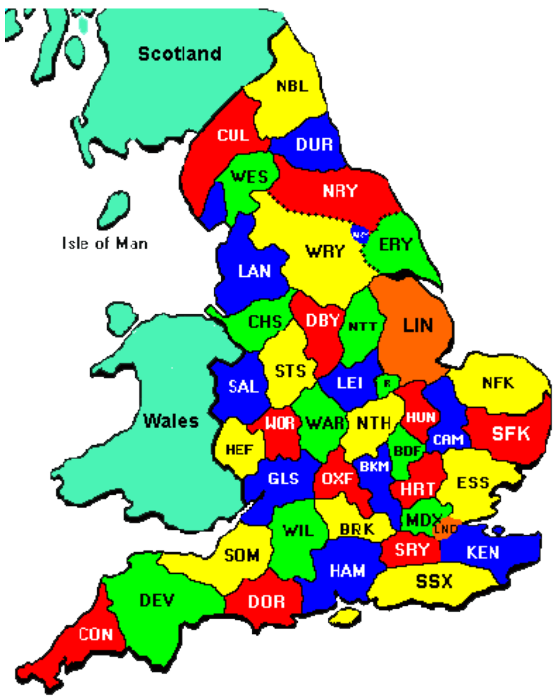
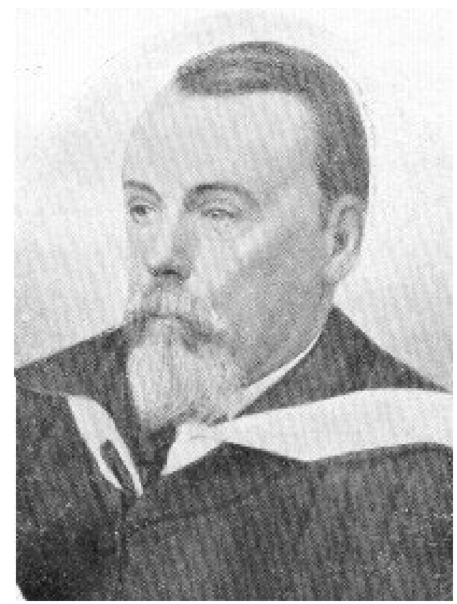
The Four Color Problem

The Four Color problem (Francis Guthrie, 1852)

Can any planar map consisting of contiguous regions be colored using four colors so that adjacent regions receive different colors?



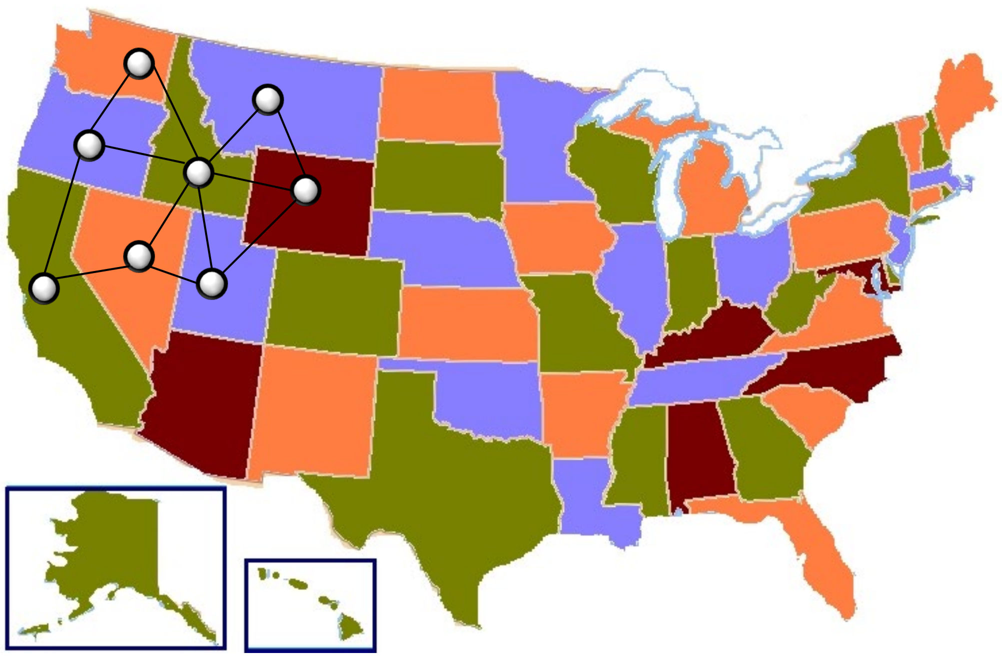


Francis Guthrie
1831-1899

The Four Color Theorem

The Four Color theorem (Appel, Haken, 1976)

Any planar map consisting of contiguous regions can be colored using four colors so that adjacent regions receive different colors.



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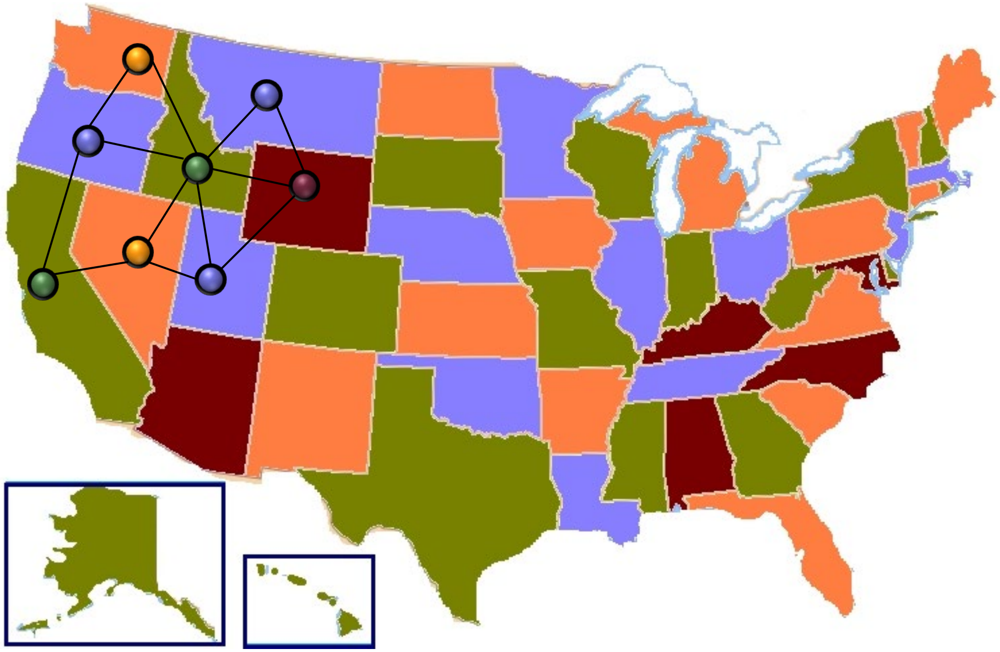
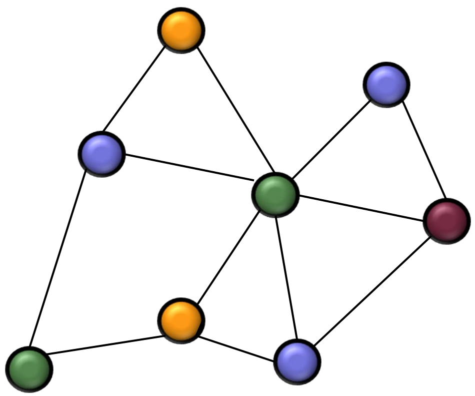


Graph coloring

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is the vertex set, and $E(G)$ is the edge set, consisting of some pairs of elements of $V(G)$.

The complete graph K_t has t vertices and an edge joining every pair of vertices.

A map $c : V(G) \to \{1, 2, \ldots, k\}$ is a k-coloring of G if $c(u) \neq c(v)$ for every $\{u, v\} \in E(G)$.
A graph G is k-colorable if it admits a k-coloring.
The chromatic number $\chi(G)$ is the minimum k such that G is k-colorable.



4-coloring

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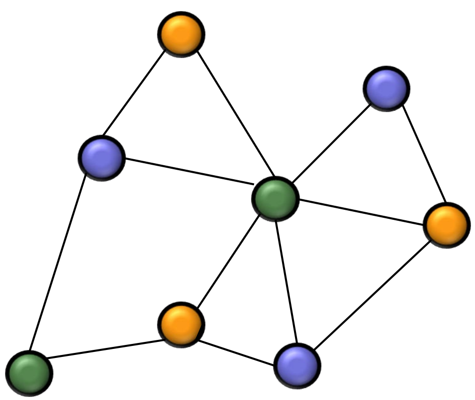

3-coloring

Graph coloring

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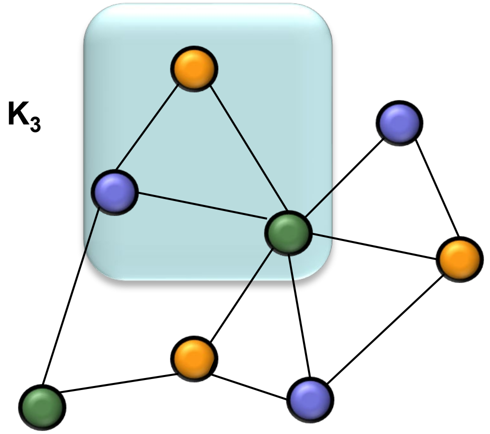

$\chi(G)=3$

Graph coloring

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The Four Color theorem (Appel, Haken, 1976)

Every planar graph is 4-colorable.

The Four Color theorem

The Four Color theorem (Appel, Haken, 1976)

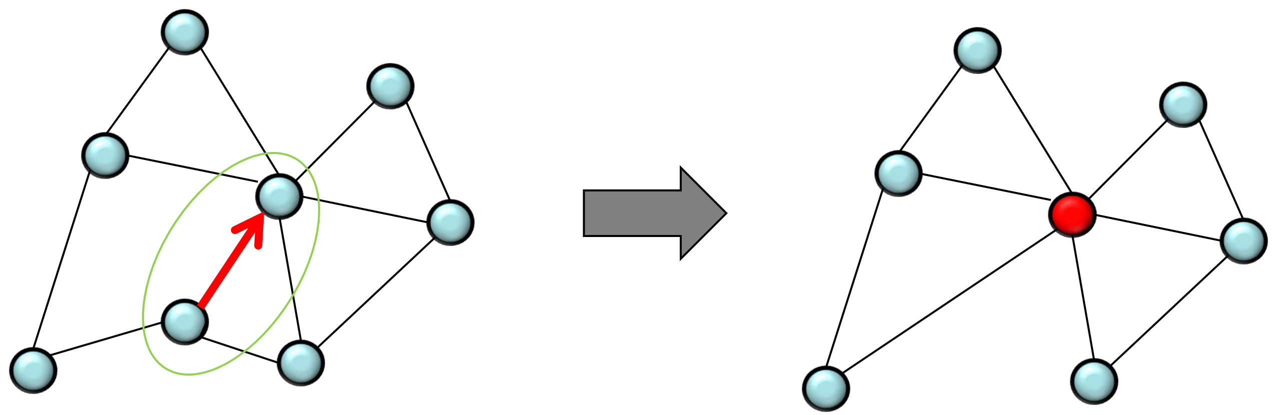
Every planar graph is 4-colorable.

- The proof of the Four-Color Theorem is computer-generated, and can not be reasonably considered to be human-readable.
- The Four-Color Theorem has equivalent reformulations in terms of vector cross products, Lie algebras, divisibility, Temperley-Lieb algebras, etc.

Minors

Minors

A graph H is a minor of a graph G if H can be obtained from G by repeated contraction of edges and deletion of vertices and edges.



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Theorem (Kuratowski, Wagner)

A graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.

Theorem (Wagner, 1937)

The Four Color theorem implies that every graph with no K_5 minor is 4-colorable.

Hadwiger's conjecture

Minors

A graph H is a minor of a graph G if H can be obtained from G by repeated contraction of edges and deletion of vertices and edges.

Conjecture (Hadwiger, 1943)

For every integer $t \geq 1$, every graph with no K_{t+1} minor is t-colorable.

Known results

- Easy for $t \leq 2$
- $t = 3$: Hadwiger, 1943
- $t = 4$: Appel and Haken, 1976
- $t = 5$: Robertson, Seymour and Thomas, 1993

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Known results

Challenge 0: The Four Color Problem

Find a computer-free proof of the Four Color Theorem

- $t = 5$: Robertson, Seymour and Thomas, 1993

Hadwiger's conjecture

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Known results

Challenge 1: The next case of Hadwiger's conjecture

Show that every graph with no K_7 minor is 6-colorable.

- $t = 5$: Robertson, Seymour and Thomas, 1993

Hadwiger's conjecture

$f(t)$-colorable.

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For every integer $t \geq 1$, every graph with no ~~K_{t+1}~~ H_t minor is t-colorable.

Hadwiger's conjecture

Conjecture (Hadwiger, 1943)

-“coarsely”-colorable.

For every integer $t \geq 1$, every graph with no K_{t+1} minor is t-colorable.

1. Improving the chromatic number

Conjecture (Hadwiger, 1943)

For every integer $t \geq 1$, every graph with no K_{t+1} minor is t-colorable.

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Hadwiger's conjecture

Conjecture (Hadwiger, 1943)

For every integer $t \geq 1$, every graph with no K_{t+1} minor is t-colorable.

Special case

Let G be a graph with no independent set of size three. If $|V(G)| = n$ then $\chi(G) \geq n/2$, and so Hadwiger's conjecture implies that G has a $K_{\lceil n/2 \rceil}$ minor.

Challenge 2: Independence number two

Show that for some $c > 1/3$ every graph with n vertices and no independence set of size three contains a K_t with $t \geq cn$.

Weak Hadwiger's conjecture

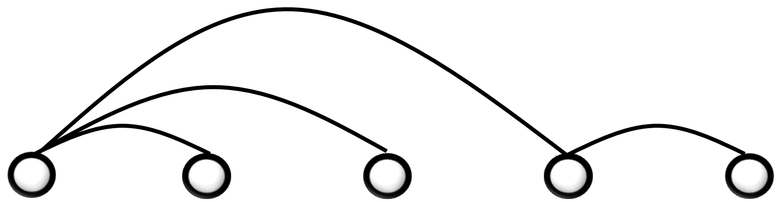
Challenge 3: Weak Hadwiger's Conjecture

There exists $C > 0$ such that $\chi(G) \leq Ct$ for every graph G with no K_t minor.

Degeneracy

A graph G is d-degenerate if there exists an ordering of $V(G)$ in which every vertex is preceded by at most d neighbors.

Every d-degenerate graph is $(d+1)$-colorable.



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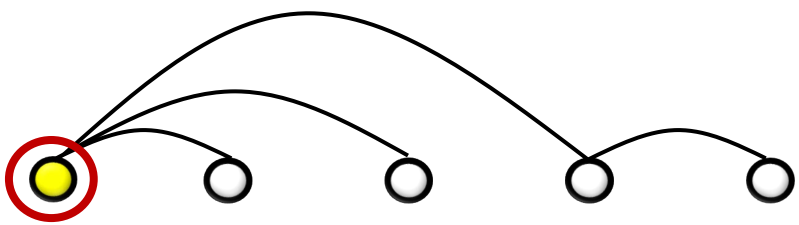

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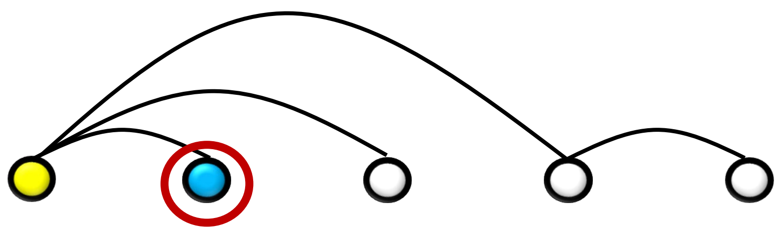

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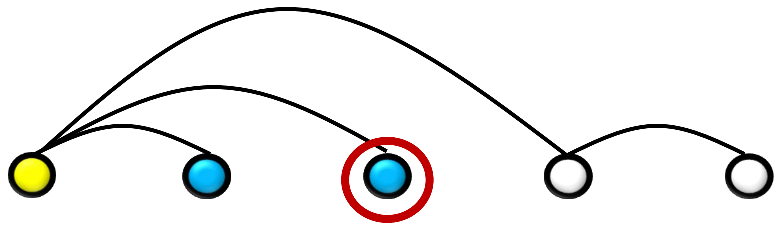

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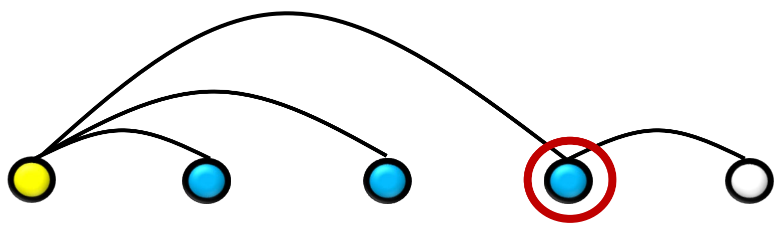

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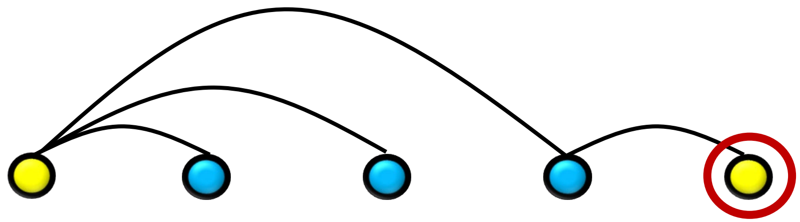

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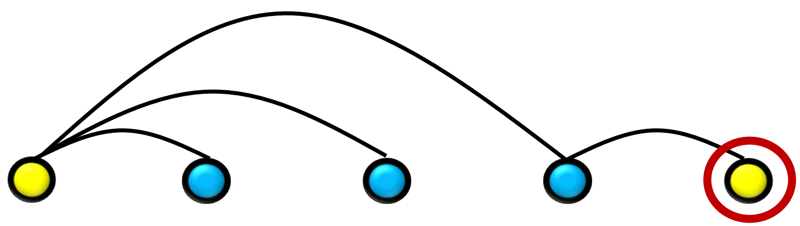

For $t = 1, 2, 3$ every graph with no K_{t+1}-minor is $(t-1)$-degenerate.

Degeneracy

Theorem (Kostochka, Thomason, '80s)

Every graph with no K_t minor is $O(t\sqrt{\log t})$-degenerate.

This bound is tight: Extremal examples include dense random graphs.

"Hadwiger's counter-conjecture" (per Reed and Seymour, 1997)

For large t there exists graphs G with no K_t minor and $\chi(G) = \Omega(t\sqrt{\log t})$

Breaking the degeneracy barrier

Theorem (Kostochka, Thomason, 1980s)

Every graph with no K_t minor is $O(t(\log t)^{1/2})$-colorable.

Theorem (N.,Song, 2019+)

Every graph with no K_t minor is $O(t(\log t)^{0.354})$-colorable.

Theorem (Delcourt, Postle, 2021+)

Every graph with no K_t minor is $O(t \log \log t)$-colorable.

2. Obtaining a sparser minor

Conjecture (Hadwiger, 1943)

For every integer $t \geq 1$, every graph with no ~~K_t~~ H_t minor is t-colorable.

H-Hadwiger's conjecture

H-Hadwiger's conjecture (Seymour 2017)

For every graph H on $t+1$ vertices, every graph with no H minor is t-colorable.

Known for

- $H = K_{2,t}$ (Woodall, 2001)
- $H = K_{s,t}$ if $t > C(s \log s)^3$ (Kostochka, 2013)

Theorem (N., Turcotte, 2021+)

For every $\Delta > 0$, H-Hadwiger's conjecture holds for sufficiently large planar bipartite graphs H with maximum degree at most Δ.

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Bandwidth Theorem (Böttcher, Schacht and Taraz, 2009)

For every positive integer Δ and every $\gamma > 0$ there exists N such that for every planar bipartite graph H with maximum degree at most Δ and at least N vertices, if G is a graph such that $|V(G)| \geq |V(H)|$ and minimum degree at least $(1+\gamma)\frac{|V(G)|}{2}$ then G contains a subgraph isomorphic to H.

H-Hadwiger's conjecture

H-Hadwiger's conjecture (Seymour 2017)

For every graph H on $t+1$ vertices, every graph with no H minor is t-colorable.

Challenge 4:

Prove H-Hadwiger's conjecture for large classes of graphs H. For which graphs H every graph G with no H minor is $(|V(H)|-2)$-degenerate?

3. Coloring improperly

Conjecture (Hadwiger, 1943)

-“coarsely”-colorable.

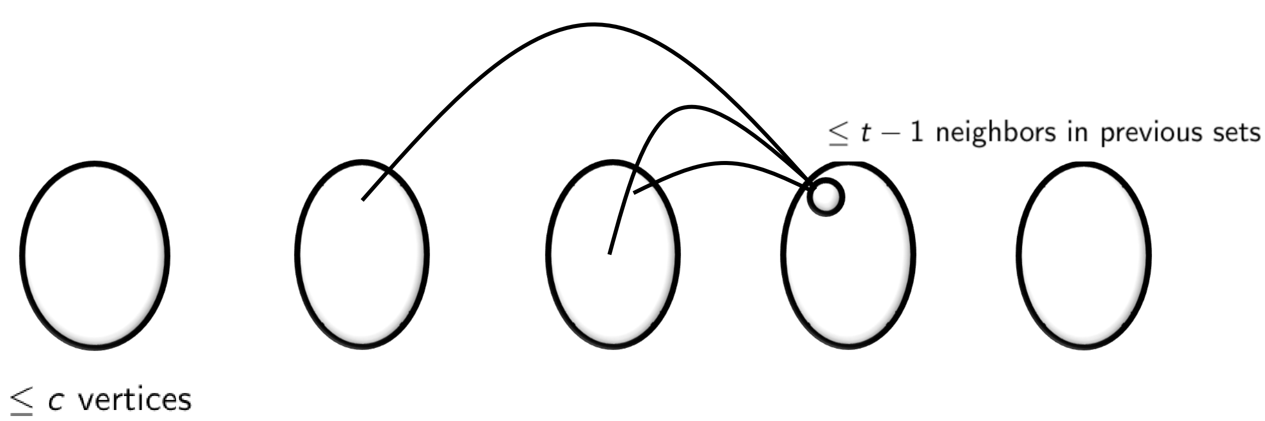
For every integer $t \geq 1$, every graph with no K_{t+1} minor is ~~t-colorable~~.

Coarse colorings

The class of graphs $\mathcal{F}$ is coarsely t-colorable if there exists c such that the vertices of every graph $G \in \mathcal{F}$ can be (improperly) colored in t colors so that every connected monochromatic subgraph of G has at most c vertices.

Theorem (Dvorák, N., 2018+)

The class of graphs with no K_{t+1}-minor is coarsely t-colorable. (In fact, it is coarsely $(t-1)$-degenerate.)



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Theorem (Dvorák, N., 2018+)

The class of graphs with no K_{t+1}-minor is coarsely t-colorable. (In fact, it is coarsely $(t-1)$-degenerate.)

Challenge 5:

Which minor-closed classes of graphs $\mathcal{F}$ are coarsely t-colorable? Which are coarsely $(t-1)$-degenerate?

$\leq c$ vertices

Summary

Conjecture (Hadwiger, 1943)

For every integer $t \geq 1$, every graph with no K_{t+1} minor is t-colorable.

- Hadwiger's conjecture is an ambitious generalization of the Four-Color Theorem.
- Although it appears to be out of reach in full generality,
- we are getting closer little by little and continue learning about connections between structure and coloring along the way.

Theorem (Dvorák, N., 2018+)

The class of graphs with ...
coarsely $(t-1)$...

...ficiently large planar

...fact, it is

For every $\Delta > 0$, ...
bipartite graphs H with ma...

Theorem (Delcourt, Postle, 2021+)

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Thank you!