

Random algebraic dynamics in complex dimension 2.

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Outline

- ▶ X is a compact complex surface = complex analytic manifold of dimension 2 (real dimension 4).
- ▶ $\Gamma \leq \text{HolDiff}(X)$ group of holomorphic diffeomorphisms of X .

Goal

Describe the dynamics of Γ on X .

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We only consider “large” subgroups.

Definition

Γ is **non-elementary** if there exists $f, g \in \Gamma$ with positive topological entropy, and such that $\langle f, g \rangle \leq \Gamma$ is a free subgroup.

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- ▶ Classification of **Γ -invariant measures**.
- ▶ **Asymptotic distribution** of orbits.

Properties of non-elementary groups Γ

Theorem

If X is a compact complex surface and $\Gamma \leq \text{HolDiff}(X)$ is non-elementary, then X is projective and Γ acts by algebraic transformations.

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- ▶ most examples come from constructions from algebraic geometry: such examples are **rare** and typically live in finite dimensional families. We will only present a few of them.

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- ▶ most examples come from constructions from algebraic geometry: such examples are **rare** and typically live in finite dimensional families. We will only present a few of them.
- ▶ $\text{Aut}(X)$ (hence Γ) is typically **discrete**.

Linear examples: Abelian and Kummer surfaces

Example 1: affine maps on complex tori

- ▶ $X = \mathbb{C}^2/L$, L a lattice.
- ▶ Any automorphism of X comes from an affine map on \mathbb{C}^2 commuting with L .
- ▶ Example: $E = \mathbb{C}/\Lambda$ an elliptic curve

$$X = E \times E = \mathbb{C}^2/\Lambda \times \Lambda$$

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- ▶ The natural $\mathrm{GL}_2(\mathbb{Z})$ action on \mathbb{C}^2 commutes with $L = \Lambda \times \Lambda$ so it induces an action on X .

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The dynamics of such automorphism groups is well understood:

- ▶ orbit closures (Guivarc'h-Starkov 2004, Muchnik 2005)
- ▶ distribution of orbits (Bourgain-Furman-Lindenstrauss-Mozes 2008, Benoist-Quint 2010, He-De Saxcé 2020, He-Lakrec-Lindenstrauss 2022, etc.)

Linear examples: Abelian and Kummer surfaces

Example 1': Kummer construction

- ▶ Consider the involution $\eta(x, y) = (-x, -y)$ on $X = E \times E$.
- ▶ Take the quotient X/η and desingularize $Y := \widehat{X/\eta}$
- ▶ η commutes with the action of $\mathrm{GL}_2(\mathbb{Z})$, so $\mathrm{GL}_2(\mathbb{Z})$ descends to a non-elementary group $\Gamma \subset \mathrm{Aut}(Y)$.

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- ▶ This construction can be generalized to more general finite quotients $Y = \widehat{X/G}$: this gives the family of **Kummer groups** (linear maps in disguise!).
 - ▶ They often appear as **exceptional cases** in our results (cf. Lattès examples in 1D holomorphic dynamics)

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- ▶ Markov showed that all positive solutions can recursively be obtained from $(1, 1, 1)$ by permuting the coordinates and applying the involutive transformation

$$\sigma_z : (x, y, z) \mapsto (x, y, 3xy - z)$$

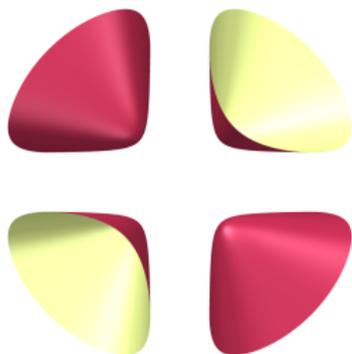
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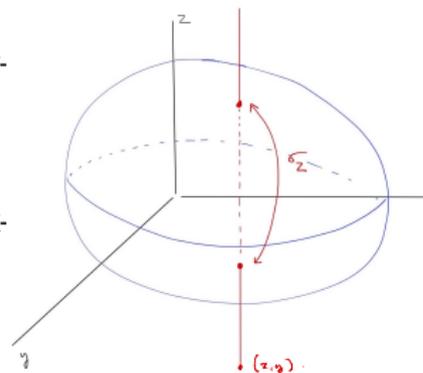
Similarly we can define involutions σ_x, σ_y .
Thus, the Markoff equation defines a surface S in \mathbb{C}^3 with a large group of automorphisms

Groups generated by involutions

The geometric mechanism behind the Markoff transformation is very simple:

- ▶ S is of degree 2 w.r.t each coordinate
- ▶ so given $(x, y) \in \mathbb{C}^2$ there is an involution σ_z exchanging the 2 points in the corresponding fiber

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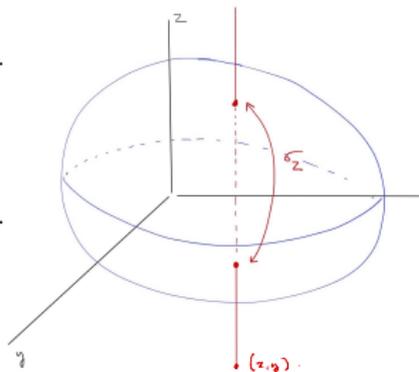
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Similar examples play an important role in low dimensional topology (mapping class group actions on varieties of characters of $SL(2, \mathbb{C})$ representations), leading to families of surfaces like :

$$x^2 + y^2 + z^2 = xyz + k \quad (\text{punctured torus})$$

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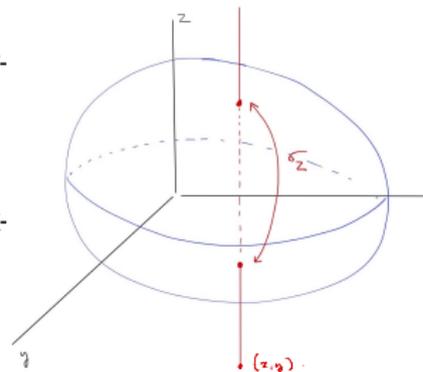
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This group also appears in the study of the monodromy of the Painlevé VI differential equation, etc. **Arises in many, many different contexts!**



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$$a_{222}x^2y^2z^2 + a_{221}x^2y^2z + \cdots + a_{100}x + a_{010}y + a_{001}z + a_{000} = 0$$

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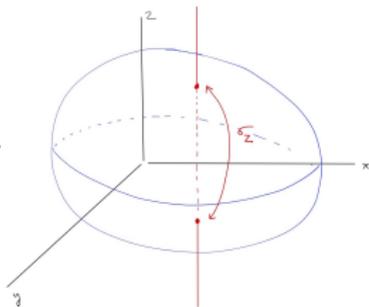
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- ▶ This is the **Wehler family** \mathcal{W} of (2,2,2) surfaces. $X \in \mathcal{W}$ is parameterized by $[a_{222} : a_{221} : \cdots : a_{000}] \in \mathbb{P}^{26}(\mathbb{C})$.

- ▶ Any $X \in \mathcal{W}$ admits three coordinate involutions $\sigma_x, \sigma_y, \sigma_z$.



Wehler's (2,2,2) surfaces

Proposition

For generic $X \in \mathcal{W}$

1. X is a smooth K3 surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
2. $\Gamma = \langle \sigma_x, \sigma_y, \sigma_z \rangle$ is a non-elementary subgroup of $\text{Aut}(X)$ isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.
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Remarks

- ▶ X admits a Γ -invariant volume form $i\Omega \wedge \bar{\Omega}$, where

$$\Omega = \frac{dx \wedge dy}{\partial_z P} = \frac{dy \wedge dz}{\partial_x P} = \frac{dz \wedge dx}{\partial_y P}$$

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- ▶ If X is defined over a subfield of \mathbb{C} (like \mathbb{Q} , $\bar{\mathbb{Q}}$ or \mathbb{R}) then so does Γ .
- ▶ The Kummer “linear” examples embed as singular (hence non-generic) Wehler surfaces: gives an algebraic way of deforming linear automorphisms of tori.

Main results for the Wehler family

Theorem (orbit closures, $K = \mathbb{C}$)

There exists a dense open subset $\Omega \subset \mathcal{W}$ such that for every $X \in \Omega$, there exists:

- ▶ a Γ -invariant finite set F
- ▶ a Γ -invariant totally real analytic surface Y (possibly singular)

such that for every $x \in X$:

- (i) either $x \in F$ (and its orbit is finite)
- (ii) or $\overline{\Gamma \cdot x}$ is a union of components of Y
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- ▶ F is the maximal finite invariant set.
- ▶ The typical situation in case (ii) is that X is defined over \mathbb{R} and Y is a union of components of $X(\mathbb{R})$.
- ▶ The open set Ω is essentially explicit and locally the complement of a real analytic subset.

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Theorem (Equidistribution, $K = \mathbb{R}$)

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- (i) either $x \in F$
- (ii) or $\Gamma \cdot x$ is equidistributed in $X'(\mathbb{R})$, a union of components of $X(\mathbb{R})$.

Here equidistribution is in the following sense: pick (γ_n) a sequence of independent random variables equidistributed among σ_x , σ_y and σ_z , then

$$\frac{1}{n} \sum_{k=1}^n \delta_{\gamma_k \cdots \gamma_1 x} \xrightarrow[n \rightarrow \infty]{} \text{vol}_{X'(\mathbb{R})},$$

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Conjecture

Equidistribution holds for $K = \mathbb{C}$.

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Additional comments

- ▶ The orbit closure statement is purely topological but the proof relies on ergodic theory.
- ▶ The equidistribution statement is real but the proof uses holomorphic dynamics.

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- ▶ **Remainder of the talk:** put these results into their proper context and give some ideas of the arguments.

Action on cohomology / Classification of automorphisms

- ▶ First step of the analysis of a holomorphic dynamical system on X : action on cohomology.

$$f : X \rightarrow X \text{ induces } f^* \in \text{GL}(H_{DR}^*(X, \mathbb{C})).$$

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- ▶ Γ is non-elementary iff $\Gamma^* \leq \mathrm{GL}(H^2(X, \mathbb{C}))$ is non-elementary.

Classification of automorphisms: elliptic maps

- ▶ If f is elliptic then f^* is of finite order, equivalently some iterate f^k is in $\text{Aut}(X)^0$: time 1 map of a holomorphic vector field.
- ▶ Examples: translations on an Abelian surface or projective linear maps on \mathbb{P}^2 .
- ▶ The dynamics of f is easy to describe.

Classification of automorphisms: parabolic maps

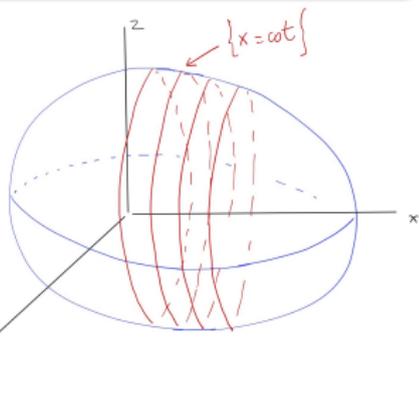
Theorem (Gizatullin, 1980)

If f is parabolic then $\|(f^*)^n\| \asymp n^2$ and f preserves a fibration by genus 1 curves $X \rightarrow B$.

If X is not a torus, f_B has finite order, so replacing f by f^k it preserves the fibers and acts upon them as a translation, with a transversal “shearing” property.

Example

On a Wehler (2,2,2) surface, $g = \sigma_y \circ \sigma_z$ preserves the pencil of curves $\{x = C^{\text{st}}\}$. These are (2,2) curves in $\mathbb{P}^1 \times \mathbb{P}^1$ hence elliptic and g acts upon them like a translation.



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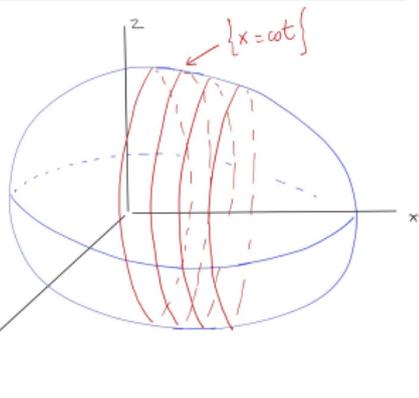
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- ▶ The dynamics and ergodic theory of such transformations can be described completely

Classification of automorphisms: loxodromic maps

For loxodromic f , put $\lambda_1(f) = \lim_{n \rightarrow \infty} \|(f^*)^n\|^{1/n}$: dynamical degree of f .

Theorem (Gromov 1979, Yomdin 1987)

The topological entropy of f is equal to $\log \lambda_1$.

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Thus the dynamics of a loxodromic automorphism is chaotic and does not admit a simple description.

Classification of automorphisms: loxodromic maps

For loxodromic f , put $\lambda_1(f) = \lim_{n \rightarrow \infty} \|(f^*)^n\|^{1/n}$: dynamical degree of f .

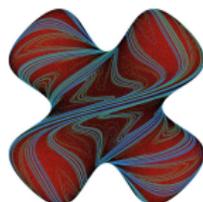
Theorem (Gromov 1979, Yomdin 1987)

The topological entropy of f is equal to $\log \lambda_1$.

Thus the dynamics of a loxodromic automorphism is chaotic and does not admit a simple description.

Example

On a $(2,2,2)$ surface, $f = \sigma_x \circ \sigma_y \circ \sigma_z$ is loxodromic of topological entropy $\log(9 + 4\sqrt{5})$.



Loxodromic Wehler automorphisms
(pictures by Curt McMullen)

Finite orbits for non-elementary groups

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Example: linear maps on tori

Let Γ be a non-elementary group of linear maps acting on $A = \mathbb{C}^2/L$. Then $\text{Per}(\Gamma)$ is the set of torsion points of A .

- ▶ More generally the same holds for a **Kummer group**: the set of Γ -periodic points is dense (hence Zariski dense).

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Theorem (non-Zariski density of finite orbits, $K = \overline{\mathbb{Q}}$)

Let Γ be a non-elementary group acting on X . Assume:

- ▶ X and Γ are both defined over some number field.
- ▶ Γ contains a parabolic element.

Then if $\text{Per}(\Gamma)$ is Zariski-dense, then (X, Γ) is a Kummer group.

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Comment

- ▶ There always exists a maximal invariant algebraic curve. So the theorem says that outside this curve there are finitely many periodic points.
- ▶ Corollary: if there is no invariant algebraic curve, $\text{Per}(\Gamma)$ is finite.

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Let Γ be a non-elementary group acting on X . Assume:

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Then $\text{Per}(\Gamma)$ is finite.

Comment

- ▶ The proof is by a specialization argument: reduction to $K = \overline{\mathbb{Q}}$.

Finite orbits for non-elementary groups

Back to the description of orbit closures in the Wehler family.

Theorem (orbit closures, $K = \mathbb{C}$)

There exists a dense open subset $\Omega \subset \mathcal{W}$ such that for every $X \in \Omega$, there exists:

- ▶ a Γ -invariant finite set F
- ▶ a Γ -invariant totally real analytic surface Y (possibly singular)

such that for every $x \in X$:

- (i) either $x \in F$ (and its orbit is finite)
- (ii) or $\overline{\Gamma \cdot x}$ is a union of components of Y
- (iii) or $\overline{\Gamma \cdot x} = X$.

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- ▶ $\text{Per}(\Gamma) \subset \text{Per}(\langle f, g \rangle) \subset \text{Per}(f) \cap \text{Per}(g)$, and there is no reason for these sets to have a large intersection.
- ▶ So one should expect

$\text{Per}(f) \cap \text{Per}(g)$ infinite \Rightarrow something special happens

“unlikely intersection problem” (cf. Zannier).

Finite orbits for non-elementary groups

Theorem (Baker-DeMarco, Yuan-Zhang 2011)

If f and g are polynomials in $\mathbb{C}[X]$, then

$$\begin{aligned} \text{Per}(f) \cap \text{Per}(g) \text{ infinite} &\Rightarrow J_f = J_g \\ &\Rightarrow f \text{ and } g \text{ are dynamically related.} \end{aligned}$$

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Theorem (D.-Favre 2016)

If f and g are polynomial automorphisms of \mathbb{C}^2 with non-trivial dynamics and $|\text{Jac}(f)| \neq 1$, then

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All these results use tools from arithmetic geometry (hence reduction to $K = \overline{\mathbb{Q}}$). In our setting the key step is to use arithmetic equidistribution to produce a Γ -invariant measure, which we can classify thanks to the presence of parabolic elements in Γ .

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- ▶ Fix ν a probability measure on Γ such that $\langle \text{Supp}(\nu) \rangle = \Gamma$. For simplicity assume ν is symmetric and finitely supported.
Example: $\nu = \frac{1}{3}(\delta_{\sigma_x} + \delta_{\sigma_y} + \delta_{\sigma_z})$ on a (2,2,2) surface.

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- ▶ We now have a **random dynamical system** on X and our goal is to analyze the **asymptotic distribution of trajectories** $\gamma_n \cdots \gamma_1 x$ on X .

Stationary and invariant measures

- ▶ **Basic principle:** the asymptotic distribution of sample paths $\gamma_n \cdots \gamma_1 x$ is described by **stationary measures**

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A probability measure μ on X is stationary if $\mu = \nu * \mu = \int f_* \mu d\nu(f)$.

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Definition (Furstenberg)

A random dynamical system is **stiff** if every stationary measure is invariant.

Stiffness

Theorem (stiffness, $K = \mathbb{R}$)

Let $\Gamma \leq \text{Aut}(X)$ and assume that X and Γ are defined over \mathbb{R} . Assume that:

- ▶ Γ preserves a smooth volume on $X(\mathbb{R})$
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Conjecture

Stiffness holds for $K = \mathbb{C}$ as well.

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Step 1: real step.

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 - ▶ either μ is finitely supported (hence invariant)
 - ▶ or μ is absolutely continuous (hence invariant)
 - ▶ or an unlikely phenomenon happens “**stable directions are non-random**”.

Explanation: for $\nu^{\mathbb{N}}$ -a.e. $\omega = (\gamma_n)$ and μ -a.e. x there is a stable Oseledets direction $E^s(\omega, x)$ and the conclusion is that $E^s(\omega, x)$ does not depend on ω .

If we can show that μ is invariant in this last case, we are done.

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The proof relies on an important principle in holomorphic dynamics:
connect local dynamical information to the action on cohomology.

The logical chain is:

- stable directions → local stable manifolds
- global stable manifolds = entire curves ($\mathbb{C} \hookrightarrow X$)
- positive closed currents (Ahlfors)
- cohomology classes in $H^{1,1}(X)$

Equidistribution

Recall the equidistribution theorem for Wehler (2,2,2) surfaces:

Theorem (Equidistribution, $K = \mathbb{R}$)

There exists a dense open subset $\Omega' \subset \mathcal{W}(\mathbb{R})$ such that for every $X \in \Omega'$, there exists a Γ -invariant finite set F such that for every $x \in X(\mathbb{R})$:

- (i) either $x \in F$
- (ii) or $\Gamma \cdot x$ is equidistributed in $X'(\mathbb{R})$, a union of components of $X(\mathbb{R})$.

The conclusion is **stronger than the classification of stationary measures**: it says that limiting measures from x are of **pure type**: atomic or Lebesgue.

In other words, we have to show that if x has infinite orbit, the limiting measures starting from x give no mass to F .

Proof of equidistribution

We have to show that if x has infinite orbit, the limiting measures from x give no mass to F .

- ▶ Using ideas from Eskin-Margulis and Benoist-Quint, it is enough to show that F is “repelling on average”.
- ▶ Since there is no a priori bound on the cardinality of F we show that generically all possible periodic orbits are repelling.
- ▶ This follows from the following [uniform expansion property](#)

$$\exists c > 0, \exists n \geq 1, \forall x \in X, \forall v \in T_x M, \int \log \|df_x(v)\| d\nu^n(f) \geq c.$$

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Uniform expansion holds on an open and dense set of (real or complex) Wehler surfaces.

This relies on ergodic theory, in particular ideas from the proof of the stiffness theorem.

Thanks for your attention!