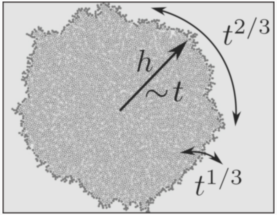
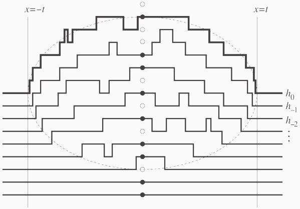
Integrable fluctuations in one-dimensional random growth

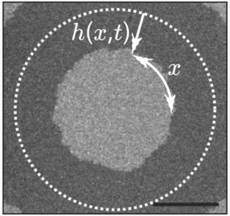
Daniel Remenik

Universidad de Chile

ICM, July 2022







Outline

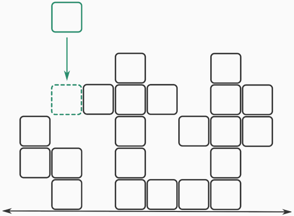
1. KPZ universality class, KPZ fixed point
2. Polynuclear growth model
3. Integrable PDEs
4. Some proof ideas

Joint work with K. Matetski and J. Quastel

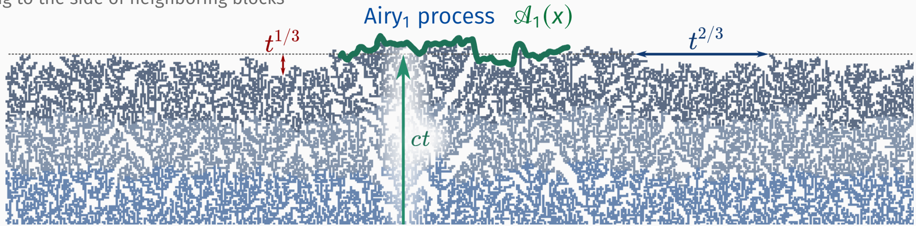
One-dimensional random growth

Ballistic deposition

Expected: $t^{-1/3}\left(h(t, t^{-2/3}x) - ct\right) \xrightarrow[t\to\infty]{} \mathscr{A}(x)$

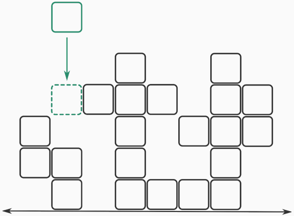


Blocks fall at rate 1 at each site,
sticking to the side of neighboring blocks



One-dimensional random growth

Ballistic deposition



Blocks fall at rate 1 at each site,
sticking to the side of neighboring blocks

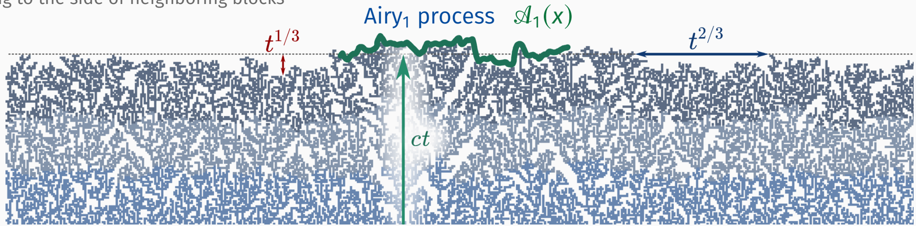
Expected: $t^{-1/3}\big(h(t, t^{-2/3}x) - ct\big) \xrightarrow[t\to\infty]{} \mathscr{A}(x)$

(Conjectural) member of the KPZ universality class

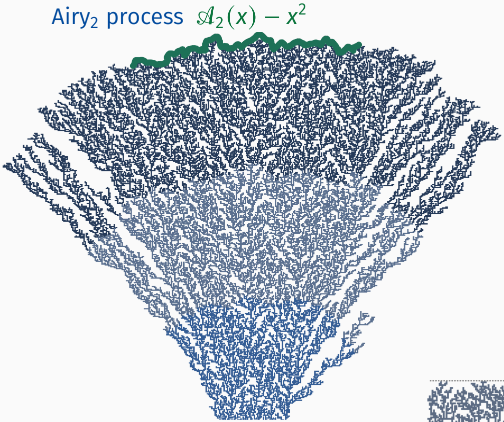
- Stoch. 1-d interface growth (Eden model)
- Stoch. PDEs (KPZ equation)
- Interacting particle systems (exclusion processes)
- Directed random polymers/last passage percolation ......

All have an associated height function.

Same expected behavior



One-dimensional random growth



Expected: $t^{-1/3}(h(t, t^{-2/3}x) - ct) \xrightarrow[t\to\infty]{} \mathscr{A}(x)$

(Conjectural) member of the KPZ universality class

Stoch. 1-d interface growth (Eden model)

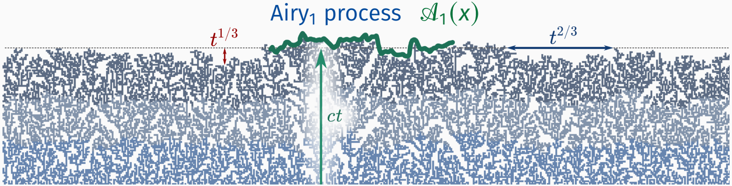
Stoch. PDEs (KPZ eqn.)

Interacting particle systems (exclusion processes)

Directed random polymers/last passage percolation ......

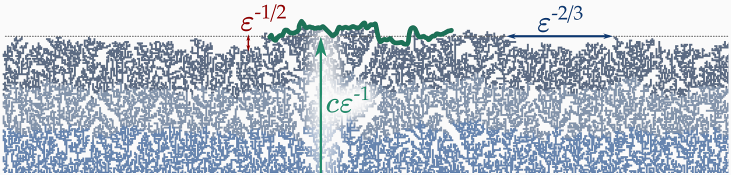
All have an associated height function

Same expected behavior



KPZ universality conjecture

KPZ 1:2:3 scaling: $h(t,x) \longmapsto \varepsilon^{1/2}\big(h(c_1\varepsilon^{-3/2}t, c_2\varepsilon^{-1}x) - C_\varepsilon t\big)$



KPZ universality conjecture

Conjecture (or definition):

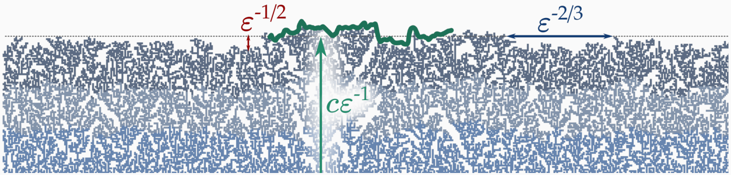
For all models in the KPZ universality class, the *height function* $h(t,x)$ satisfies

(assuming $\varepsilon^{1/2}h(0,2\varepsilon^{-1}x) \longrightarrow \mathfrak{h}_0$ suitably),

$$\varepsilon^{1/2}\left[h(c_1\varepsilon^{-3/2}t, c_2\varepsilon^{-1}x) - C_\varepsilon t\right] \longrightarrow \mathfrak{h}(t,x)$$

for some universal limit $\mathfrak{h}(t,x)$ (as a process in t and x)

KPZ 1:2:3 scaling: $h(t,x) \longmapsto \varepsilon^{1/2}\left(h(c_1\varepsilon^{-3/2}t, c_2\varepsilon^{-1}x) - C_\varepsilon t\right)$



KPZ universality conjecture

Conjecture (or definition):

For all models in the KPZ universality class, the *height function* $h(t,x)$ satisfies

(assuming $\varepsilon^{1/2}h(0,2\varepsilon^{-1}x) \longrightarrow \mathfrak{h}_0$ suitably),

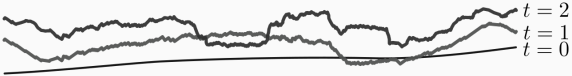
$$\varepsilon^{1/2}\Big[h(c_1\varepsilon^{-3/2}t, c_2\varepsilon^{-1}x) - C_\varepsilon t\Big] \longrightarrow \mathfrak{h}(t,x)$$

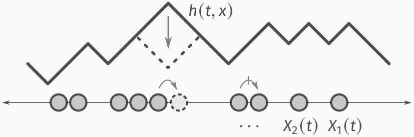
for some universal limit $\mathfrak{h}(t,x)$ (as a process in t and x)

KPZ 1:2:3 scaling: $h(t,x) \longmapsto \varepsilon^{1/2}\big(h(c_1\varepsilon^{-3/2}t, c_2\varepsilon^{-1}x) - C_\varepsilon t\big)$

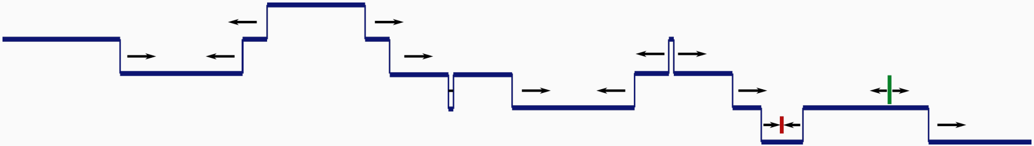
$\mathfrak{h}(t,x)$ is known as the KPZ fixed point (invariant under KPZ 1:2:3 rescaling)

First constructed in [Matetski-Quastel-R '17] as the 1:2:3 scaling limit of TASEP





The polynuclear growth process (PNG)



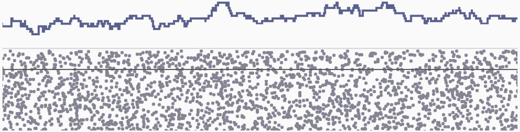
Height function $h : \mathbb{R} \longrightarrow \mathbb{Z} \cup \{-\infty\}$ [Gates-Westcott ’95, Prähofer-Spohn ’02]

Down jumps  move to the right at speed 1, up jumps  move to the left at speed 1

Up/down jump  nucleations at space-time Poisson pt. proc., rate 2

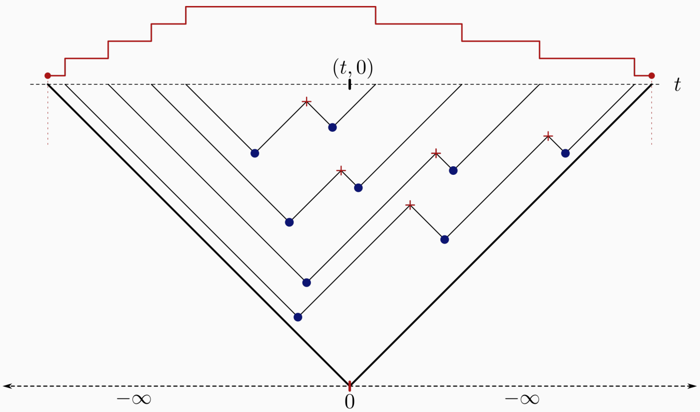
Expanding islands annihilate on contact 

Independent Poisson up and down jumps, rate 1, are invariant



[Simulation by Patrik Ferrari]

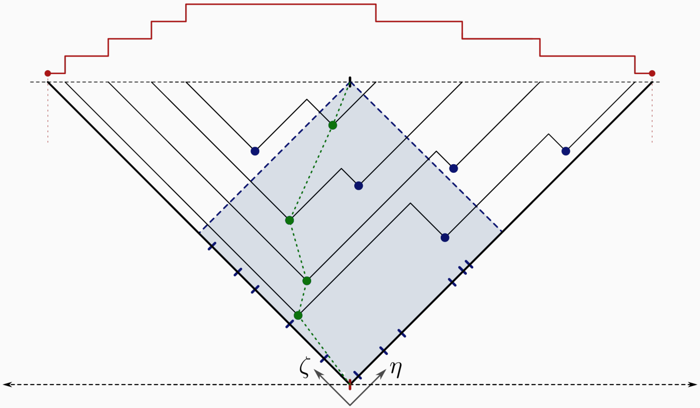
Droplet/narrow wedge initial condition



Blue dots are creation events | Red crosses are annihilations

Initial condition is $\mathfrak{h}_0(0) = 0$, $\mathfrak{h}_0(x) = -\infty$ if $x \neq 0$ narrow wedge

Droplet/narrow wedge initial condition

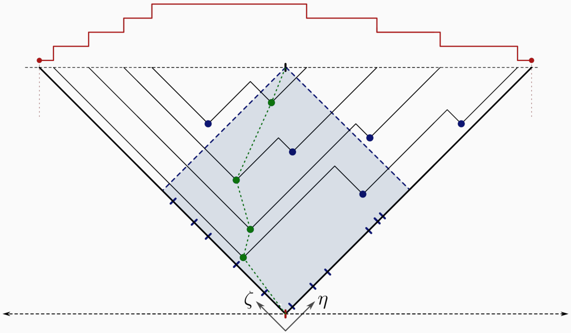


Blue dots are creation events | Red crosses are annihilations

Initial condition is $\mathfrak{d}_0(0) = 0,\ \mathfrak{d}_0(x) = -\infty$ if $x \neq 0$ narrow wedge

$$h(t,x;h_0) = \sup_{y\in\mathbb{R}} \left\{ h(t,x;\mathfrak{d}_y) + h_0(y) \right\} \qquad \text{(Poissonian LPP)}$$

Droplet/narrow wedge initial condition



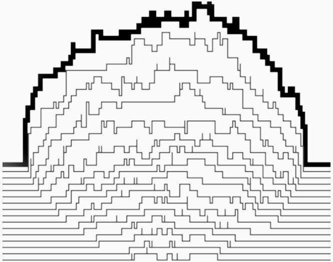
$N \sim \text{Poisson}[t^2]$ points inside the square, η-ζ order defines $\sigma \in S_N$

$h(t,0)$ is the length of longest increasing subsequence in σ

[Baik-Deift-Johansson '99] $N^{-1/6}(\ell_N - 2\sqrt{N}) \longrightarrow$ Tracy-Widom GUE

Narrow wedge/droplet: $t^{-1/3}(h(t, t^{2/3}x) - 2t) \longrightarrow \mathcal{A}_2(x) - x^2$ [Prähofer-Spohn '02, Johansson '03]

Airy$_2$ process, TW-GUE marginals



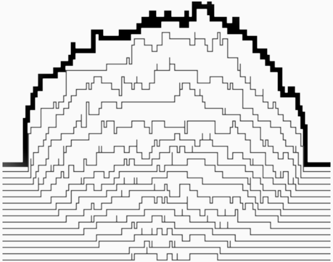
Narrow wedge/droplet: $t^{-1/3}(h(t, t^{2/3}x) - 2t) \longrightarrow \mathcal{A}_2(x) - x^2$ [Prähofer-Spohn '02, Johansson '03]

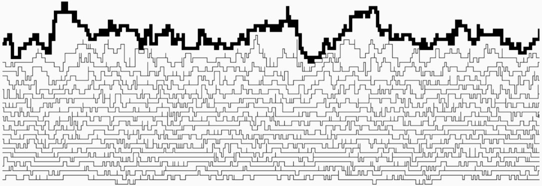
Airy$_2$ process, TW-GUE marginals

Flat ($h_0 \equiv 0$): $t^{-1/3}(h(t, t^{2/3}x) - 2t) \longrightarrow \mathcal{A}_1(x)$ [Borodin-Ferrari-Sasamoto '08]

Airy$_1$ process, TW-GOE marginals

(symmetrized random permutations [Baik-Rains '01])





Similar results for exclusion processes, LPP [Johansson, Borodin, Ferrari, Prähofer, Spohn, '01-'10]

Many extensions for related processes during the last decade

The KPZ fixed point

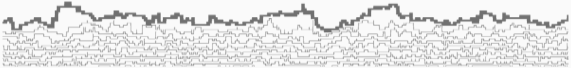
Thm:

For the PNG model, suppose $\varepsilon^{1/2} h(0, \varepsilon^{-1} x) \longrightarrow \mathfrak{h}_0$ suitably. Then the limit

$$\mathfrak{h}(t,x) = \lim_{\varepsilon \longrightarrow 0} \varepsilon^{1/2} \Big[h(\varepsilon^{-3/2} t, \varepsilon^{-1} x) - 2\varepsilon^{-3/2} t \Big]$$

exists, and it defines a Markov process in the space of upper-semicont. functions.

- $\mathfrak{h}(t,x)$ is the KPZ fixed point constructed in [Matetski-Quastel-R '17] as the 1:2:3 scaling limit of TASEP
- Result follows from [Dauvergne-Virág '21] or [Matetski-Quastel-R '22]
- It has explicit transition probabilities given as Fredholm determinants $\rightsquigarrow$ integrability
- Locally Brownian in x, Brownian motion is invariant (modulo global height)
- Variational description [Dauvergne-Ortmann-Virág '18]
- Fixed $t = 1$, narrow wedge/flat $\longrightarrow$ $\text{Airy}_2/\text{Airy}_1$ results
- Recent progress in universality



Integrability of the KPZ fixed point

Classical integrability of the Gaussian ensembles:

F_{GOE} and F_{GUE} can be expressed in terms of a solution of the Hastings-McLeod soln. to the Painlevé II eqn. [Tracy-Widom ’94, ’96]

$$q'' = rq + 2q^3 \qquad \text{with} \quad q(x) \sim \mathrm{Ai}(x), \;\; x \longrightarrow \infty$$

Explicitly:

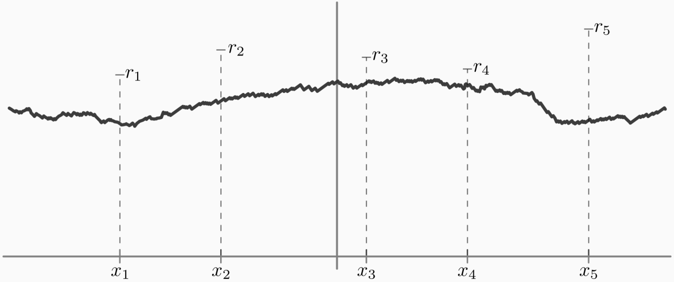
$$F_{\mathrm{GUE}}(r) = \exp\left(-\int_r^\infty \mathrm{d}u\,(u-r)q^2(u)\right) \qquad \text{(narrow wedge)}$$

$$F_{\mathrm{GOE}}(r) = \exp\left(-\tfrac{1}{2}\int_r^\infty \mathrm{d}u\,q(u)\right)F_{\mathrm{GUE}}(r)^{1/2} \qquad \text{(flat)}$$

Many other results available for special KPZ models, mostly restricted to narrow-wedge initial conditions

Integrability of the KPZ fixed point

Let $\mathbf{F}(t, x_1, \ldots, x_n, r_1, \ldots, r_n) = \mathbb{P}_{\mathfrak{h}_0}\big(\mathfrak{h}(t, x_i) \leq r_i,\ i = 1, \ldots, n\big)$



Integrability of the KPZ fixed point

Let $\mathbf{F}(t, x_1, \ldots, x_n, r_1, \ldots, r_n) = \mathbb{P}_{\mathfrak{h}_0}\big(\mathfrak{h}(t, x_i) \leq r_i,\ i = 1, \ldots, n\big)$

For general (deterministic) initial data $\mathfrak{h}_0$:

Thm: [Quastel-R ’19]

In the case $n = 1$, $\mathbf{F}_r(t, x) = \mathbb{P}(\mathfrak{h}(t, x) \leq r)$,

$\phi = \partial_r^2 \log \mathbf{F}$ satisfies the

Kadomtsev–Petviashvili (KP-II) equation

$$\partial_r\big(\partial_t\phi + \phi\,\partial_r\phi + \tfrac{1}{12}\partial_r^3\phi\big) + \tfrac{1}{4}\partial_x^2\phi = 0.$$

KP: natural 2-d extension of the KdV eqn. for long waves in shallow water

Integrability of the KPZ fixed point

Let $\mathbf{F}(t, x_1, \ldots, x_n, r_1, \ldots, r_n) = \mathbb{P}_{\mathfrak{h}_0}\big(\mathfrak{h}(t, x_i) \le r_i,\ i = 1, \ldots, n\big)$

For general (deterministic) initial data $\mathfrak{h}_0$:

Thm: [Quastel-R '19]

In the case $n = 1$, $\mathbf{F}_r(t, x) = \mathbb{P}(\mathfrak{h}(t, x) \le r)$,
$\phi = \partial_r^2 \log \mathbf{F}$ satisfies the
Kadomtsev–Petviashvili (KP-II) equation

$$\partial_r\big(\partial_t\phi + \phi\,\partial_r\phi + \tfrac{1}{12}\partial_r^3\phi\big) + \tfrac{1}{4}\partial_x^2\phi = 0.$$

Multipoint version: $\partial_r \log \mathbf{F} = \operatorname{tr}(\mathbf{Q})$
with the $n \times n$ matrices $\mathbf{Q}$ and $\mathbf{q} = \partial_r\mathbf{Q}$ solving the
matrix KP equation

$$\partial_t\mathbf{q} + \tfrac{1}{2}\partial_r\mathbf{q}^2 + \tfrac{1}{12}\partial_r^3\mathbf{q} + \tfrac{1}{4}\partial_x^2\mathbf{Q} + \tfrac{1}{2}(\mathbf{q}\,\partial_x\mathbf{Q} - \partial_x\mathbf{Q}\mathbf{q}) = 0.$$

KP: natural 2-d extension of the KdV eqn. for long waves in shallow water

Integrability of the KPZ fixed point

Let $\mathbf{F}(t, x_1, \ldots, x_n, r_1, \ldots, r_n) = \mathbb{P}_{\mathfrak{h}_0}\big(\mathfrak{h}(t, x_i) \leq r_i,\ i = 1, \ldots, n\big)$

For general (deterministic) initial data $\mathfrak{h}_0$:

Thm: [Quastel-R '19]

In the case $n = 1$, $\mathbf{F}_r(t, x) = \mathbb{P}(\mathfrak{h}(t, x) \leq r)$,
$\phi = \partial_r^2 \log \mathbf{F}$ satisfies the
Kadomtsev–Petviashvili (KP-II) equation

$$\partial_r\big(\partial_t \phi + \phi\,\partial_r \phi + \tfrac{1}{12}\partial_r^3 \phi\big) + \tfrac{1}{4}\partial_x^2 \phi = 0.$$

Multipoint version: $\partial_r \log \mathbf{F} = \mathrm{tr}(\mathbf{Q})$
with the $n \times n$ matrices $\mathbf{Q}$ and $\mathbf{q} = \partial_r \mathbf{Q}$ solving the
matrix KP equation

$$\partial_t \mathbf{q} + \tfrac{1}{2}\partial_r \mathbf{q}^2 + \tfrac{1}{12}\partial_r^3 \mathbf{q} + \tfrac{1}{4}\partial_x^2 \mathbf{Q} + \tfrac{1}{2}(\mathbf{q}\,\partial_x \mathbf{Q} - \partial_x \mathbf{Q}\,\mathbf{q}) = 0.$$

KP: natural 2-d extension of the KdV eqn. for long waves in shallow water

Cor: $\mathfrak{h}_0 = \text{flat}$, $\phi(t, x, r) = 4^{2/3} t^{-2/3} \psi(4^{1/3} t^{-1/3} r)$, $\psi = \frac{1}{2}(q' - q^2)$ (Miura's transform)

$\implies q'' = 2q^3 + rq$ Painlevé II equation

$\implies \mathbf{F}(1, 0, r) = e^{-\frac{1}{2}\int_r^\infty (q(s) + (r-s) q(s)^2)\, ds} \rightsquigarrow$ recovers TW-GOE

Similar for narrow-wedge/TW-GUE

PNG Fredholm determinant formula

One-point distributions: $\mathbb{P}_{h_0}(h(t,x) \leq r) = \det\left(I - K^{h_0}_{t,x}\right)_{\ell^2(\mathbb{Z}_{>r})}$

(multipoint formulas are similar)

$$\det(I-K)_{\ell^2(\mathbb{Z}_{>r})} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{x_1>r} \cdots \sum_{x_n>r} \det\left[K(x_i,x_j)\right]_{i,j=1}^{n}$$

PNG Fredholm determinant formula

One-point distributions: $\mathbb{P}_{h_0}\big(h(t,x) \le r\big) = \det\big(I - K^{h_0}_{t,x}\big)_{\ell^2(\mathbb{Z}_{>r})}$

(multipoint formulas are similar)

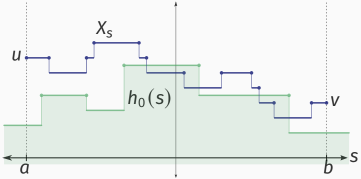
$$\det(I-K)_{\ell^2(\mathbb{Z}_{>r})} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{x_1>r} \cdots \sum_{x_n>r} \det\big[K(x_i,x_j)\big]_{i,j=1}^{n}$$

Let $\nabla f(x) = \frac{1}{2}(f(x+1) - f(x-1))$, $\Delta f(x) = f(x+1) - 2f(x) + f(x-1)$,

X_s a symmetric, rate 1, nearest neighbor RW on $\mathbb{Z}$

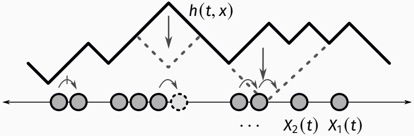
$$P^{\mathrm{hit}(h_0)}_{a,b}(u,v) = \mathbb{P}_{X_a=u}\big(X \text{ hits hypo}(h_0) \text{ on } [a,b],\ X_b = v\big)$$

$$\mathscr{T}_{a,b} = e^{a\Delta} P^{\mathrm{hit}(h_0)}_{a,b} e^{-b\Delta}(u,v).$$

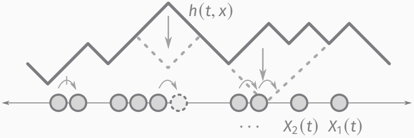


Then $\boxed{K^{h_0}_{t,x} = e^{-2t\nabla + x\Delta}\,\mathscr{T}_{x-t,x+t}\,e^{2t\nabla + x\Delta}}$

The formula was first obtained as the scaling limit of a similar formula for discrete time TASEP with parallel update via *biorthogonalization* [Matetski-R '22]



The formula was first obtained as the scaling limit of a similar formula for discrete time TASEP with parallel update via *biorthogonalization* [Matetski-R '22]



Alternative pf.: check directly that $F_r(t,x)$ satisfies the *Kolmogorov backw. eqn.*

$$(\partial_t - \mathscr{L})\det(I-K) = \det(I-K)\,\mathrm{tr}[(I-K)^{-1}(\partial_t - \mathscr{L})K], \text{ so we need } \boxed{\partial_t K = \mathscr{L} K}$$

$$\partial_t K = e^{-2t\nabla + x\Delta}\big(-2\nabla\mathcal{T}^{h_0}_{x-t,x+t} + 2\mathcal{T}^{h_0}_{x-t,x+t}\nabla\big)e^{2t\nabla+x\Delta}, \quad \mathscr{L}K = e^{-2t\nabla+x\Delta}\mathscr{L}\mathcal{T}^{h_0}_{x-t,x+t}e^{2t\nabla+x\Delta}$$

Key fact: $g(s) := X_s$ is invariant under the PNG dynamics modulo height shifts

By the symmetry of the problem and integration by parts,

$$\begin{aligned}\mathscr{L}_{h_0}P^{\mathrm{hit}(h)}_{x-t,x+t}(u,v) &= \mathscr{L}_{h_0}\mathbb{P}_{g(x-t)=u}\big(g \text{ hits } \mathrm{hypo}(h),\ g(x+t)=v\big)\\ &= \mathscr{L}^*_g\mathbb{P}_{g(x-t)=u}\big(g \text{ hits } \mathrm{hypo}(h_0),\ g(x+t)=v\big)\\ &= 2P^{\mathrm{hit}(h_0)}_{x-t,x+t}\nabla(u,v) - 2\nabla P^{\mathrm{hit}(h_0)}_{x-t,x+t}(u,v)\end{aligned}$$

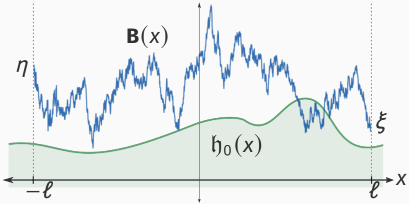
KPZ fixed point formula

$$K^{h_0}_{t,x} = e^{-2t\nabla + x\Delta}\mathcal{T}_{x-t,x+t}e^{2t\nabla+x\Delta}.$$

After ε rescaling: $X_s \longrightarrow B_s$,

$$e^{x\Delta} \longrightarrow e^{x\partial^2} \text{ (heat kernel)}$$

$$e^{2t\nabla} \approx e^{\frac{t}{3}\partial^3 + \text{shift}}$$



$\mathbf{K}^{\text{hypo}(\mathfrak{h}_0)}_{\text{Brownian}}$ $\sim$ asymptotic probab. for a BM to hit hypograph$(\mathfrak{h}_0)$

Brownian scattering operator

For the KPZ fixed point,

$$\mathbb{P}_{\mathfrak{h}_0}(\mathfrak{h}(t,x) \le r) = \det\left(\mathbf{I} - e^{-\frac{t}{3}\partial^3 - x\partial^2}\mathbf{K}^{\text{hypo}(\mathfrak{h}_0)}_{\text{Brownian}}\, e^{\frac{t}{3}\partial^3 + x\partial^2}\right)_{L^2((r,\infty))}$$

Leads to KP / matrix KP equations...

In the narrow-wedge case the [Baik-Deift-Johansson '99] proof started with the Toeplitz determinant formula $F_r(s) := \mathbb{P}_{\mathfrak{d}_0}(h(s,0) \le r) = \det\big(I_{i-j}(2s)\big)_{i,j=0,\dots,r-1}$

(I_k modified Bessel functions of the first kind)

From this it can be extracted that $\alpha_r^2 = 1 - \dfrac{F_{r+1}F_{r-1}}{F_r^2}$ satisfies [Periwal-Shevitz '90, Borodin '01 ...]

$$-s(1-\alpha_r^2)(\alpha_{r+1} + \alpha_{r-1}) = r\alpha_r$$ discrete Painlevé II equation

In the narrow-wedge case the [Baik-Deift-Johansson '99] proof started with the Toeplitz determinant formula $F_r(s) := \mathbb{P}_{\mathfrak{d}_0}(h(s,0) \leq r) = \det\big(I_{i-j}(2s)\big)_{i,j=0,\dots,r-1}$

(I_k modified Bessel functions of the first kind)

From this it can be extracted that $\alpha_r^2 = 1 - \dfrac{F_{r+1}F_{r-1}}{F_r^2}$ satisfies [Periwal-Shevitz '90, Borodin '01 ...]

$$-s(1-\alpha_r^2)(\alpha_{r+1}+\alpha_{r-1}) = r\alpha_r$$

discrete Painlevé II equation

Thm: [Matetski-Quastel-R '22]

$F_r(t,x) = \mathbb{P}(h(t,x) \leq r)$

satisfies the 2D Toda equation

$$\tfrac{1}{4}(\partial_t^2 - \partial_x^2)\log F_r = \frac{F_{r+1}F_{r-1}}{F_r^2} - 1$$

$t > 0,\ r > \sup_{|y-x|\leq t} h_0(y)$

for any UC PNG initial condition h_0.

Flat: $g_r(t) = \log F_r(2t) - \log F_{r-1}(2t)$ satisfies $\ddot{g}_r = e^{g_{r+1}-g_r} - e^{g_r - g_{r-1}}$, the classic Toda lattice

In the narrow-wedge case the [Baik-Deift-Johansson '99] proof started with the Toeplitz determinant formula $F_r(s) := \mathbb{P}_{\mathfrak{d}_0}(h(s,0) \le r) = \det\big(I_{i-j}(2s)\big)_{i,j=0,\dots,r-1}$

(I_k modified Bessel functions of the first kind)

From this it can be extracted that $\alpha_r^2 = 1 - \dfrac{F_{r+1}F_{r-1}}{F_r^2}$ satisfies [Periwal-Shevitz '90, Borodin '01 ...]

$$-s(1-\alpha_r^2)(\alpha_{r+1}+\alpha_{r-1}) = r\alpha_r \qquad \text{discrete Painlevé II equation}$$

Thm: [Matetski-Quastel-R '22]

$F_r(t,x) = \mathbb{P}(h(t,x) \le r)$

satisfies the 2D Toda equation

$$\tfrac{1}{4}(\partial_t^2 - \partial_x^2)\log F_r = \frac{F_{r+1}F_{r-1}}{F_r^2} - 1$$

$$t > 0,\ r > \sup_{|y-x|\le t} h_0(y)$$

for any UC PNG initial condition h_0.

Multipoint version:

$F(t,\vec{x},\vec{r}) = \mathbb{P}\big(h(t,x_i) \le r_i,\ i = 1,\dots,n\big)$

$x = x_1 + \cdots + x_n,\ r = (r_1,\dots,r_n)$

$$\frac{F(t,x_1,\dots,x_n,r_1+1,\dots,r_n+1)}{F(t,x_1,\dots,x_n,r_1,\dots,r_n)} = \det Q_r$$

with Q_r satisfying the non-Abelian 2D Toda equations

$$\tfrac{1}{4}\partial_{t-x}(\partial_{t+x}Q_rQ_r^{-1}) + Q_rQ_{r-1}^{-1} - Q_{r+1}Q_r^{-1} = 0.$$

Flat: $g_r(t) = \log F_r(2t) - \log F_{r-1}(2t)$ satisfies $\ddot{g}_r = e^{g_{r+1}-g_r} - e^{g_r-g_{r-1}}$, the classic Toda lattice

In the narrow-wedge case the [Baik-Deift-Johansson '99] proof started with the Toeplitz determinant formula

$$F_r(s) := \mathbb{P}_{\mathfrak{d}_0}(h(s,0) \le r) = \det\big(I_{i-j}(2s)\big)_{i,j=0,\dots,r-1}$$

(I_k modified Bessel functions of the first kind)

From this it can be extracted that $\alpha_r^2 = 1 - \frac{F_{r+1}F_{r-1}}{F_r^2}$ satisfies [Periwal-Shevitz '90, Borodin '01 ...]

$$-s(1-\alpha_r^2)(\alpha_{r+1}+\alpha_{r-1}) = r\alpha_r$$

discrete Painlevé II equation

Thm: [Matetski-Quastel-R '22]

$$F_r(t,x) = \mathbb{P}(h(t,x) \le r)$$

satisfies the 2D Toda equation

$$\tfrac{1}{4}(\partial_t^2 - \partial_x^2)\log F_r = \frac{F_{r+1}F_{r-1}}{F_r^2} - 1$$

$$t > 0,\ r > \sup_{|y-x|\le t} h_0(y)$$

for any UC PNG initial condition h_0.

Multipoint version:

$$F(t,\vec{x},\vec{r}) = \mathbb{P}\big(h(t,x_i) \le r_i,\ i = 1,\dots,n\big)$$

$$x = x_1 + \dots + x_n,\ r = (r_1,\dots,r_n)$$

$$\frac{F(t,x_1,\dots,x_n,r_1+1,\dots,r_n+1)}{F(t,x_1,\dots,x_n,r_1,\dots,r_n)} = \det Q_r$$

with Q_r satisfying the non-Abelian 2D Toda equations

$$\tfrac{1}{4}\partial_{t-x}(\partial_{t+x}Q_rQ_r^{-1}) + Q_rQ_{r-1}^{-1} - Q_{r+1}Q_r^{-1} = 0.$$

$$\left.\begin{aligned} &\tfrac{1}{4}(\partial_t^2 - \partial_x^2)K(u,v) \overset{(!)}{=} K(u+1,v+1) - 2K(u,v) + K(u-1,v-1) \\ &Q_r = (I-K_r)^{-1}(1,1) \end{aligned}\right\} \rightsquigarrow \textit{Toda.}$$