A totally disconnected invitation to locally compact groups

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Based on works by/with:
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Introduction

Structure theory of locally compact groups

- Historical origin: Lie groups, Hilbert's fifth problem
- Abstract harmonic analysis, Haar's and Weil's theorems

Fundamentals

Every locally compact group G fits in a short exact sequence

$$\{e\} \longrightarrow G^\circ \longrightarrow G \longrightarrow G/G^\circ \longrightarrow \{e\}$$

where

- G° is a projective limit of **connected Lie groups** (Gleason 1950 – Yamabe 1953)
- G/G° is totally disconnected, hence **non-Archimedean** i.e. **compact open subgroups** form a basis of identity neighbourhoods (van Dantzig 1936)

Totally disconnected locally compact groups

The class of **tdlc groups** includes:

- discrete groups
- profinite groups
- algebraic groups over non-Archimedean local fields
- complete Kac–Moody groups over finite fields
- almost automorphism groups of trees
- Galois groups of field extensions of finite transcendence degree
- isometry groups of connected locally finite graphs

. . . but this list could be far from exhaustive.

Decomposition theory

Decomposition theory

Underlying question

In what ways does a given group decompose into *simple pieces*?

Definition

- A group G is **simple** if $G \neq \{e\}$ the only $N \triangleleft G$ are $N = \{e\}$ and $N = G$.
- A group G is **characteristically simple** if $G \neq \{e\}$ the only $\mathrm{Aut}(G)$-invariant $N \leq G$ are $N = \{e\}$ and $N = G$.

S simple group $\Rightarrow$ $S \times S$ is characteristically simple

Decomposition theory

Let G be a group. A finite chain of subgroups

$$\{e\} = G_0 \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G$$

is called:

- a **normal series** if $\forall i,\ G_i \triangleleft G$.
- a **subnormal series** if $\forall i,\ G_i \triangleleft G_{i+1}$.

Decomposition theory

Let $\{e\} \triangleleft M \triangleleft N \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ be a subnormal series.

The subquotient N/M is a **composition factor** of G if $M \neq N$ and for each normal subgroup $L \triangleleft N$ with $M \leq L \leq N$, we have $L = M$ or $L = N$.

Remark

A composition factor N/M is **simple**.

Decomposition theory

Let $\{e\} \leq M < N \leq G$ be a normal series.

The subquotient N/M is a **chief factor** of G if $M \neq N$ and for each normal subgroup L in G with $M \leq L \leq N$, we have $L = M$ or $L = N$.

Remark

A chief factor N/M is **characteristically simple**.

Decomposition theory

A **composition series** is a subnormal series involving only composition factors.
A **chief series** is a normal series involving only chief factors.

Example

k a field of characteristic $\neq 2$ and $|k| > 3$.
$G = k^2 \rtimes \mathrm{SL}_2(k)$.

$$\{e\} \leq \ k^2 \ \leq \ k^2 \rtimes \{\pm\mathrm{Id}\} \ \leq \ k^2 \rtimes \mathrm{SL}_2(k) \ = \ G$$

is a *chief* series (but not a *composition* series).

Decomposition theory

For *finite groups*, the decomposition theory is afforded by the Jordan–Hölder theorem.

For *infinite groups*, the existence of chief series fails in general.

Example

- $\mathbf{Z}$
- $\mathrm{SL}_3(\mathbf{Z})$

For *Lie groups*, the decomposition theory is afforded by the local correspondence with their Lie algebra.

Essentially chief series

Let G be a tdlc group. Focus on *closed* normal subgroups.

Theorem (Reid–Wesolek 2018)

*If G is **compactly generated**, then there is a normal series*

$$\{e\} \leq G_0 \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G,$$

where for all $i = 1, \ldots, n$, the subquotient G_i/G_{i-1} is:

- *discrete, or*
- *compact, or*
- *a chief factor.*

Example

- $G = \mathbf{Z}$
- $G = \mathbf{Q}_p^2 \rtimes \mathrm{SL}_2(\mathbf{Q}_p)$

Essentially chief series

Let G be a tdlc group.

Theorem (Reid–Wesolek 2018)

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Main features

- Finiteness
- Soft form of uniqueness (1st Isomorphism Theorem fails)

Essentially chief series

Key message

The basic building blocks of compactly generated tdlc groups:

- discrete groups
- compact groups
- characteristically simple groups.

Further research directions:

- Elementary tdlc groups (Wesolek)
- Typology of chief factors (Reid–Wesolek)

Essentially chief series

Theorem (C.–Monod 2011)

*Let G be a **compactly generated**, **characteristically simple**, tdlc group.*
If G is neither discrete, nor compact, then there is a non-discrete, compactly generated, simple tdlc group S, an integer $n \geq 1$, and a continuous injective homomorphism

$$S^n = S \times S \times \cdots \times S \to G$$

with dense range, and the image of each simple factor is closed.

Simple groups

Simple groups

Underlying principle (E. Cartan, 1936)

"It is a kind of historical law that the general properties of the simple groups have been verified first in the various groups, and afterward one has sought and found general explanations that do not require the examination of special cases."

Non-discrete, compactly generated simple tdlc groups include:

- Simple algebraic groups over local fields
- Many non-linear examples with a geometric flavor, e.g. $\mathrm{Isom}(T)^+$

Simple groups

Empirical observation

Very homogeneous spaces tend to have an 'almost simple' automorphism group.

- $\mathrm{Sym}(\mathbf{N})$ (R. Baër 1934)
- $\mathrm{Aut}(k^n) = \mathrm{GL}_n(k)$ (Jordan 1870; Moore 1893; Dickson 1901)
- $\mathrm{Isom}(\mathbf{S}^2) = \mathrm{O}(3)$ (Cartan 1928)
- $\mathrm{Homeo}(\mathbf{S}^2)$ (Ulam–von Neumann 1947)
- $\mathrm{Diff}(M)$ (Epstein 1970; Herman 1971; Thurston 1974)
- $\mathrm{Aut}\big([0,1], \lambda_{\mathrm{Leb}}\big)$ (A. Fathi 1974)

Simple groups

Key features of $G = \mathrm{Homeo}(M)$ acting on $X = M$.

Compressible dynamics

$$\exists\, O \subset X \text{ nonempty open such that}$$
$$\forall\, U \subset X \text{ nonempty open, } \exists\, g \in G \text{ with } gO \subset U.$$

Micro-supported action

$$\forall\, U \subset X \text{ nonempty open, } \exists\, g \in G^* \text{ with } \mathrm{Supp}(g) \subset U.$$

Simple groups

Proposition

Let X be a Hausdorff space and $G \leq \mathrm{Homeo}(X)$ with ***compressible dynamics*** *and a* ***micro-supported action****.*

Then G has a unique smallest nontrivial normal subgroup S. Moreover the S-action is micro-supported.

If in addition S itself has compressible dynamics, then S is simple.

Example

M closed manifold, $G = \mathrm{Diff}(M)$ (D. Epstein)

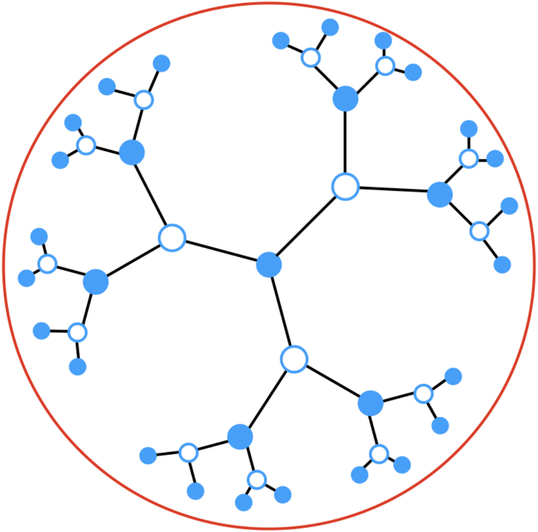
Simple groups

Key message

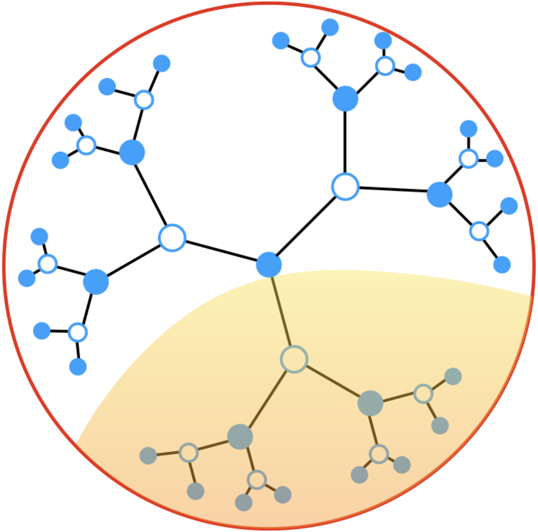
$$\text{Compressible dynamics} \quad \wedge \quad \text{Micro-supported action}$$
$$\Downarrow$$
$$\text{Almost simple group}$$

Simple tdlc groups

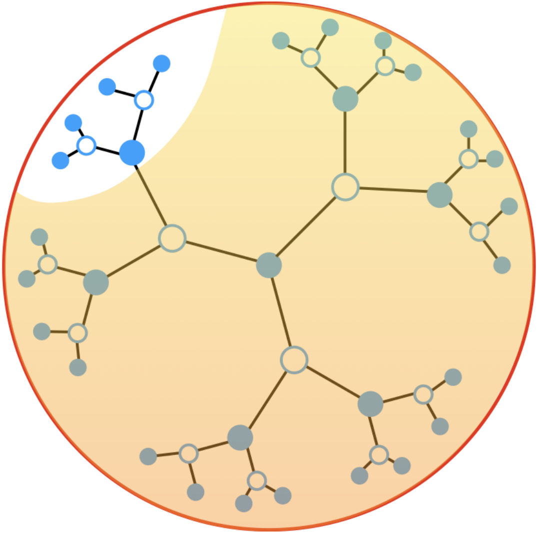
- $\mathrm{Isom}(T_d) \quad (d \geq 3)$ (J. Tits 1970)



Simple tdlc groups



Simple tdlc groups



Isom(T_d)-action on ∂T_d micro-supported, compressible dynamics.

Simple tdlc groups

Theorem (C.–Reid–W. 2017; C.–Le Boudec 2020)

Let G be a compactly generated, non-discrete, simple tdlc group.

On every ***micro-supported*** *compact G-space X, the G-action:*

- *is minimal (i.e. all orbits are dense)*
- *is strongly proximal (i.e. the closure of every G-orbit on $\mathrm{Proba}(X)$ contains a Dirac mass)*
- *has* ***compressible dynamics****.*

Micro-supported actions of G are encoded in a *local* algebraic invariant: the **centralizer lattice**.

Simple tdlc groups

Key message

For non-discrete compactly generated tdlc groups:

$$\begin{array}{l} \text{Simple} \quad \wedge \quad \text{Micro-supported action} \\ \qquad\qquad \Downarrow \\ \text{Compressible dynamics} \end{array}$$

Applications

A locally compact group G is **amenable** if on every compact G-space, there is a G-invariant probability measure.

Open problem

Can a compactly generated, non-discrete, simple tdlc group be amenable?

Corollary

Let G be a compactly generated, non-discrete, simple tdlc group.

*If G has a micro-supported action on a compact space with at least 2 points, then G is **non-amenable**.*

- Simple algebraic groups over local fields are non-amenable (but no micro-supported actions)
- Finitely generated infinite simple discrete groups can be amenable (but no compressible dynamics)

(Juschenko–Monod 2013; Nekrashevych 2018)

Applications

- Non-linearity as abstract groups
- Topological rigidity
- Abstract vs. topological simplicity
- (C.–Le Boudec) Absence of non-trivial commensurated subgroups in certain *finitely generated discrete groups*

Scale methods

Automorphisms of tdlc groups and their scale

Definition

Let $\alpha \in \mathrm{Aut}(G)$ and $U \leq G$ be compact and open. Then the scale of $\alpha \in \mathrm{Aut}(G)$ is the positive integer

$$s(\alpha) = \min\left\{[\alpha(U) : \alpha(U) \cap U] < \infty \mid U \leq G \text{ compact open}\right\}.$$

The subgroup U is minimising for α if $s(\alpha)$ is realised at U.

Example

- (H. Glöckner) If G is a p-adic Lie group and a_i are the eigenvalues of $\mathrm{ad}(\alpha)$, then $s(\alpha) = \prod\{|a_i|_p \mid |a_i|_p \geq 1\}$.
- If $G = \mathrm{Isom}(T_{q+1})$ and α_x is an inner automorphism induced by x, then $s(\alpha_x) = 1$ if x has finite orbits, and $s(\alpha_x) = q^d$ if x translates an axis by distance d.

Minimising subgroups are tidy and conversely

Theorem

Let $\alpha \in \mathrm{Aut}(G)$. *For* $U \leq G$ *compact and open, let*

$$U_+ = \bigcap_{k \geq 0} \alpha^k(U) \text{ and } U_- = \bigcap_{k \geq 0} \alpha^{-k}(U).$$

Then U *is minimising for* α *if and only if*

TA: $U = U_+U_-$ *and*

TB: $\bigcup_{n \in \mathbf{Z}} \alpha^n(U_+)$ *is closed.*

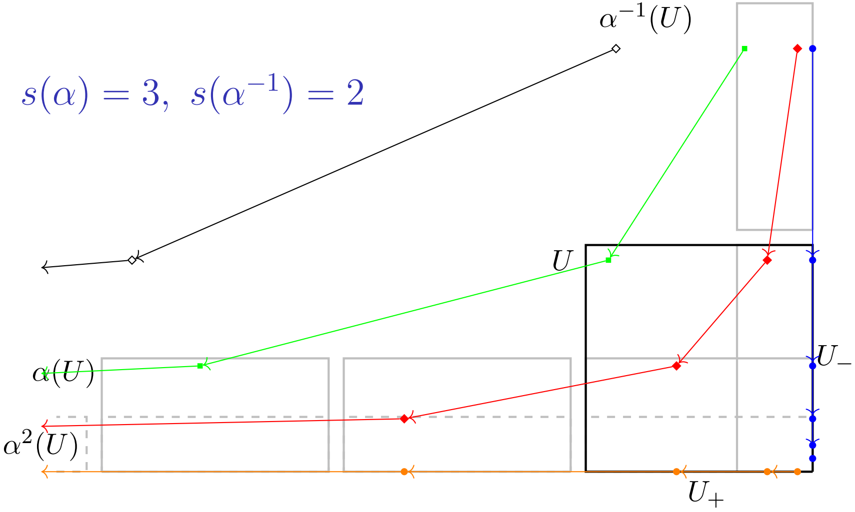
If U *satisfies these conditions, then* $s(\alpha) = [\alpha(U_+) : U_+]$.

Definition

A compact, open subgroup, U, of G satisfying **TA** and **TB** is tidy for α.

Tidy subgroups and dynamics

$$s(\alpha) = 3,\ s(\alpha^{-1}) = 2$$



Contraction groups and the scale

The contraction group is $\mathrm{con}(\alpha) := \{x \in G \mid \alpha^n(x) \to \{1\}\}$.
$\mathrm{con}(\alpha) \leq U_{--} := \bigcup_{n\in\mathbf{Z}} \alpha^n(U_-)$ but containment may be strict.

Theorem (U. Baumgartner and W.)

Suppose that U is tidy for α and set $U_0 = \bigcap_{n\in\mathbf{Z}} \alpha^n(U)$. Then

$$U_{--} = U_0 \mathrm{con}(\alpha).$$

Moreover, the closure of $\mathrm{con}(\alpha)$, denoted $\overline{\mathrm{con}}(\alpha)$, is stable under α and

$$s(\alpha^{-1}) = \Delta(\alpha^{-1}|_{U_{--}}) = \Delta(\alpha^{-1}|_{\overline{\mathrm{con}}(\alpha)}).$$

In particular, if $s(\alpha^{-1}) > 1$, then $\overline{\mathrm{con}}(\alpha)$ is not compact and $\overline{\mathrm{con}}(\alpha) \rtimes \langle \alpha \rangle$ is not unimodular.
Hence, every closed subgroup of G is unimodular if and only if G is uniscalar, *i.e.* $s(\alpha_x) = 1$ for every $x \in G$.

Applications to random walks and to ergodicity

The scale may be used to:

- prove the totally disconnected case of

Theorem (K. H. Hofmann & A. Mukherjea)

Let G be a locally compact group and μ be a probability measure on G whose support generates a dense subgroup of G. Then for every compact subset K of G, the sequence

$$f_n(K) = \sup\{\mu^{*n}(Kx) \mid x \in G\}$$

converges to zero as n tends to infinity.

- give an alternative proof of

Theorem (N. Aoki)

Let G be a tdlc group and suppose that there is an automorphism α with $\alpha \curvearrowright G$ ergodic. Then G is compact.

Flat groups of automorphisms

Definition

A group, $\mathscr{H}$, of automorphisms of G is flat if there is a compact, open $U \leq G$ that is minimising for every $\alpha \in \mathscr{H}$.
The uniscalar subgroup of $\mathscr{H}$ is

$$\mathscr{H}_u := \{\alpha \in \mathscr{H} \mid \alpha(U) = U\}\,.$$

Example

- Every cyclic group $\langle\alpha\rangle$ is flat.
- Let $G = \mathbf{Q}_p^n$ and $\mathscr{H}$ be the group of diagonal matrices $[h_j]$, $h_j \in \mathbf{Q}_p^*$, acting on $\mathbf{Q}_p^n$. Then $\mathscr{H}$ is flat and $\mathbf{Z}_p^n$ is tidy.
- Let $G = \mathrm{SL}_n(\mathbf{Q}_p)$ and $\mathscr{H}$ be the group of conjugations by diagonal elements. Then the Iwahori subgroup of $\mathrm{SL}_n(\mathbf{Z}_p)$ is minimising for $\mathscr{H}$.

Roots of flat groups

If U is tidy for $\mathscr{H}$, denote $\bigcap\{\alpha(U) \mid \alpha \in \mathscr{H}\}$ by $U_{\mathscr{H}0}$.

Definition

A root of the flat group $\mathscr{H}$ is a surjective homomorphism $\rho : \mathscr{H} \to \mathbf{Z}$ such that

$$U_\rho := \bigcap\{\alpha(U) \mid \alpha \in \mathscr{H} \text{ and } \rho(\alpha) \geq 0\} > U_{\mathscr{H}0}.$$

Denote the set of roots of $\mathscr{H}$ by Φ.

Example (ν is the valuation on $\mathbf{Q}_p$)

- Let $G = \mathbf{Q}_p^n$ and $\mathscr{H}$ be as above. The roots are
$$\rho_i([h_j]) = \nu(h_i), \quad [h_j] \in \mathscr{H},\ 1 \leq i \leq n.$$
- Let $G = \mathrm{SL}_n(\mathbf{Q}_p)$ and $\mathscr{H}$ be as above. The roots are
$$\rho_{ij}(h) = \nu(h_i h_j^{-1}), \quad 1 \leq i \neq j \leq n.$$

Flat groups and the subgroups tidy for them

Theorem

Let $\mathscr{H}$ be a flat group of automorphisms of G and suppose that $U \leq G$ is minimising for $\mathscr{H}$. Then

$$\mathscr{H}_u = \bigcap \{\ker \rho \mid \rho \in \Phi\}$$

and $\mathscr{H}/\mathscr{H}_u$ is a free abelian group.
The roots may be ordered so that $U = U_{\mathscr{H}0} U_{\rho_1} \dots U_{\rho_q}$.
For each $\rho \in \Phi$:

- *$\widetilde{U}_\rho := \bigcup \{\alpha(U_\rho) \mid \alpha \in \mathscr{H}\} \leq G$ is closed and $\mathscr{H}$-stable; and*
- *there is an integer $s_\rho > 1$ such that $\Delta(\alpha|_{\widetilde{U}_\rho}) = s_\rho^{\rho(\alpha)}$ for every $\alpha \in \mathscr{H}$.*

Example

If $G = \mathbf{Q}_p^n$ or $\mathrm{SL}_n(\mathbf{Q}_p)$, then s_{ρ_i} and $s_{\rho_{ij}}$ equal p.

Criteria for groups of automorphisms to be flat

Theorem

Let $\mathscr{H} \leq \mathrm{Aut}(G)$. *Then*

1. $\mathscr{H}$ *is flat if it is finitely generated and nilpotent*
2. $\mathscr{H}$ *is virtually flat if it is polycyclic.*

An application

Shalom & W., *Geom. Funct. Anal.*, **23** (2013) 1631–1683, uses the two last theorems to prove parts of a conjecture of G. Margulis and R. Zimmer that commensurated subgroups of certain S-arithmetic groups are commensurate with S'-arithmetic subgroups for some $S' \subseteq S$.