

The long way from mathematics to mathematics education

**How educational endeavor may change
one's vision of mathematics
& of its development**

Anna Sfard

The University of Haifa, Israel

ICM, section 19

11 Jun 2022

Defining mathematics ?!

Why?



Mark Kac

I cannot define **this creature**, but I **recognize** it when I see it!

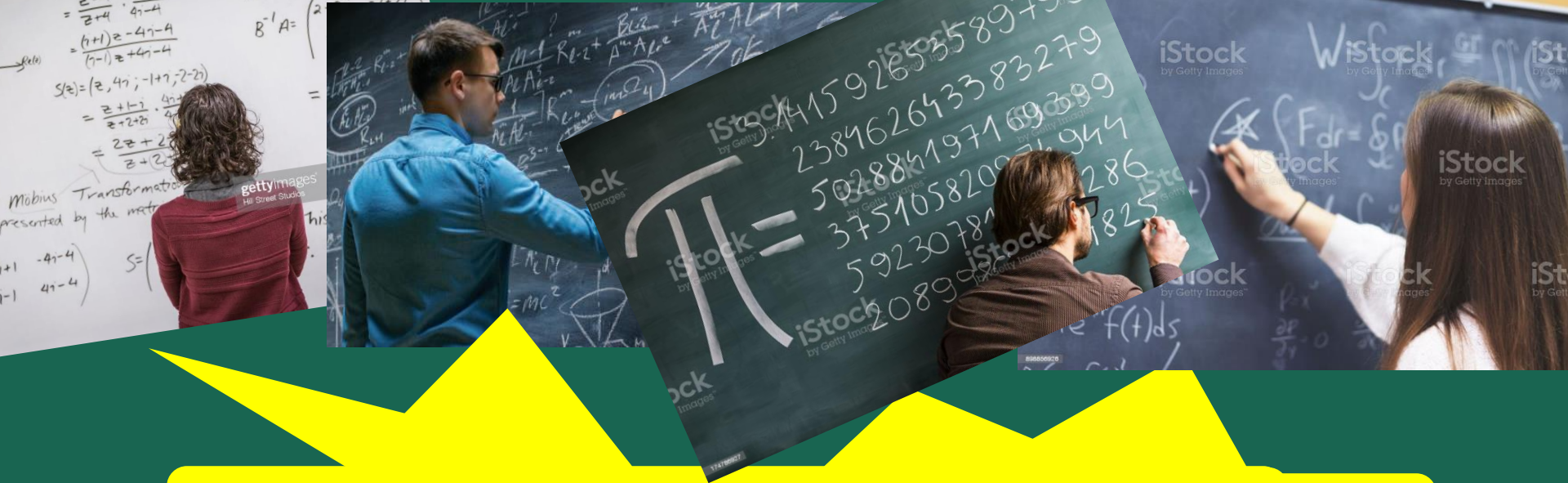
Defining mathematics ?!

Yes, we can



Mark Kac

Can't we just **do** it,
learn it, **teach** it it?



mathematicians

math teachers



Defining mathematics



Mark Kac

Yes, we can

!

Can't we just **do** it,
learn it, **teach** it it?

No, we can't

How you **teach** depends on how you think about **what you are teaching** & **how it may be learned**

What is mathematics?

- Just like **biology** is a study of **living things** (plants, animals)
- and as **physics** is a study of **material objects** (moving bodies, light, etc.)

so is
mathematics a
study of
**mathematical
objects**

But what are **mathematical objects**?

Perhaps this is because of the complexity of this question that Bertrand Russell famously stated:



Bertrand Russell

Mathematics can be described “as a subject in which we never know what we are talking about, nor whether what we are saying is true.”



But what are **mathematical objects**?

What are
mathematical objects
(MOs)
and where do they
come from?

Plan of action

1. What is mathematical object (MO)?

From dualist to monist view

2. What is mathematics?

The discursive view

3. Why does it matter?

Educational implications

Plan of action

1. What is mathematical object (MO)?

From dualist to monist view

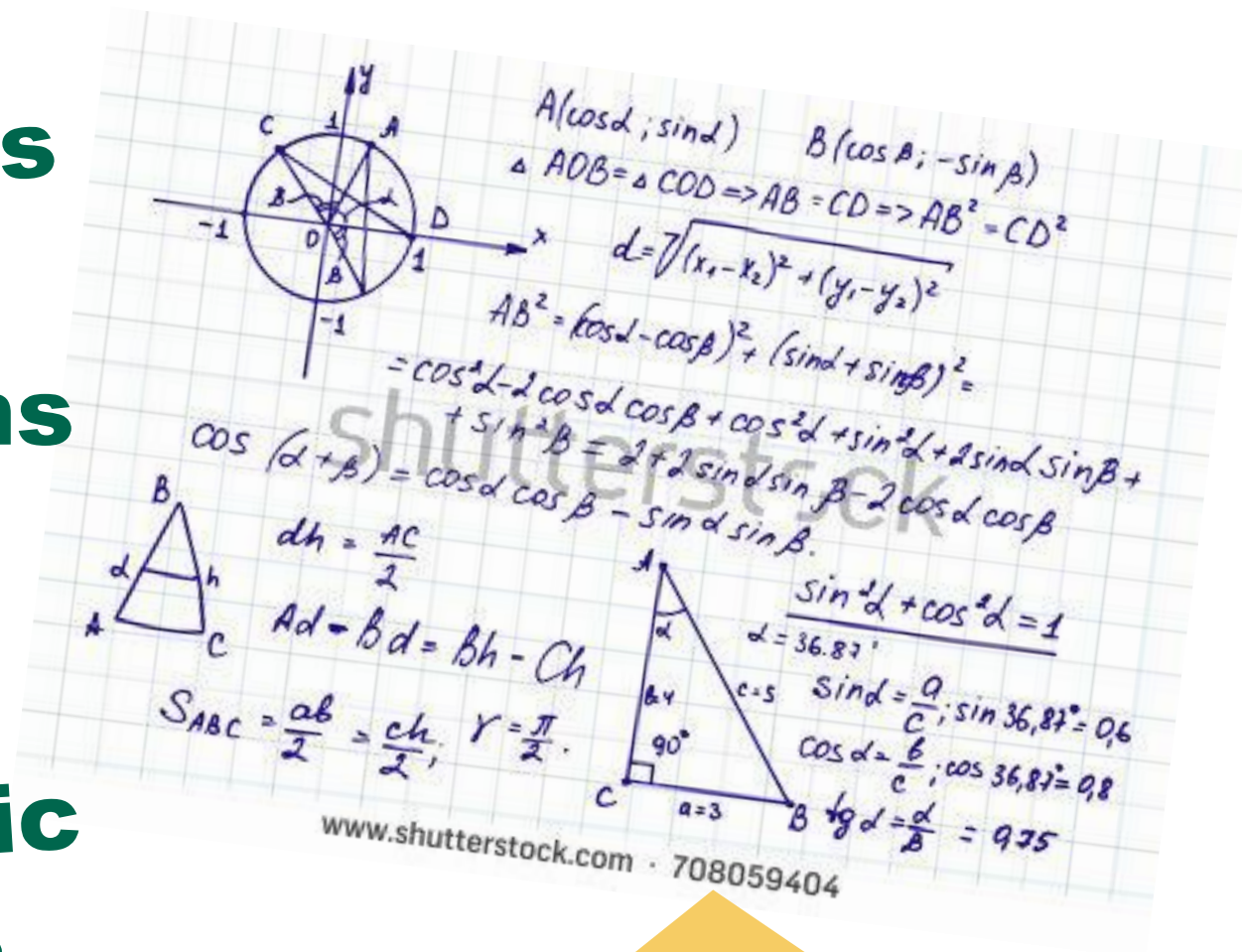
2. What is mathematics?

The discursive view

3. Why does it matter?

Educational implications

numbers
functions
sets
geometric
figures



Aren't they simply **the signs** we write while talking about them?

$$(x + 1)(x - 1) = x^2 - 1$$

Why the equals sign?

What is the same?

~~Expressions?~~

Represent the
same **functions**

The **numbers**
you get when you
substitute
numbers for x

What you write on paper/screen

$$(x + 1)(x - 1) = x^2 - 1$$

sign

equals sign?

What's the same?

mathematical object

~~Expressions?~~

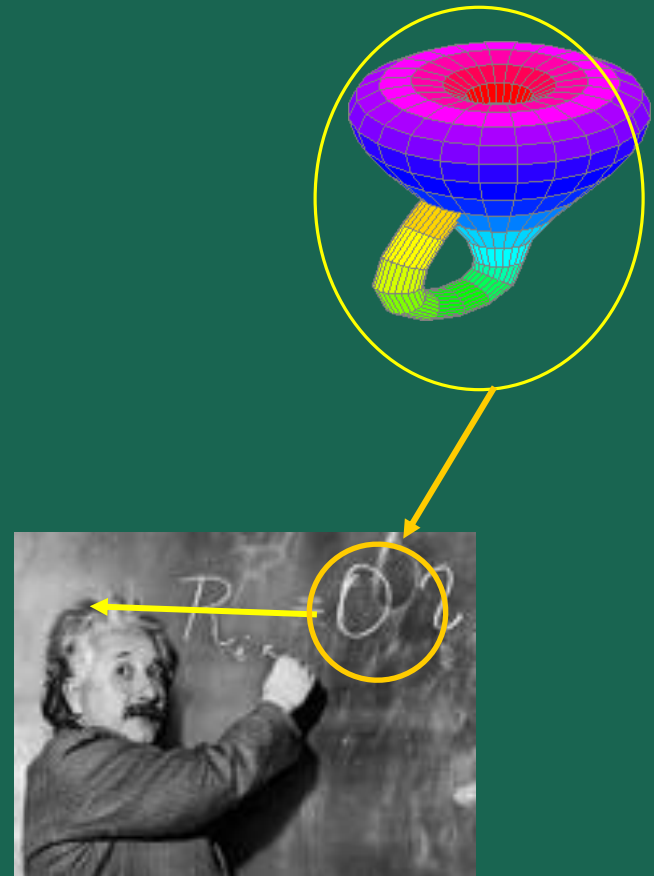
The **numbers**
you get when you
substitute
numbers for x

Represent the
same **functions**

What is mathematical object?

Platonist answer

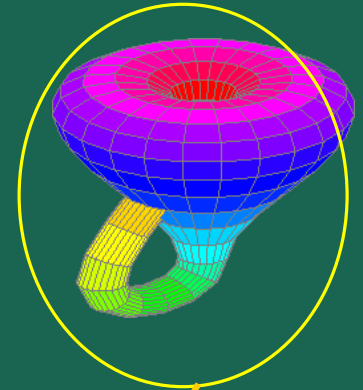
Mathematical objects are **entities that exist in the world** independently of human mind (just like physical objects, such as **rocks** or **animals**)



What is mathematical object?

Take 1:
Plat

Mathematical signs
are mere
material avatars
of independently
existing (non-material)
objects



The relation between signs and MOs

Platonism & constructivism

dualist

signs

MOs

represent

August 6, 2022

the fractions
 $1/2$, $4/8$ & 0.5
represent
the same number

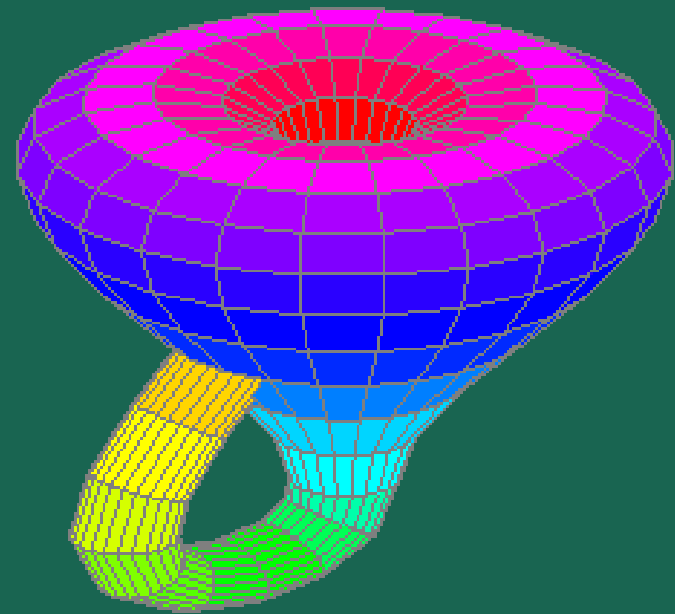
“Everything considered,
mathematicians should have the
courage of their most profound
convictions and thus affirm that
**mathematical forms indeed
have an existence that is
independent of the mind
considering them....”**



René Thom (1923-2002)

What is mathematical object?

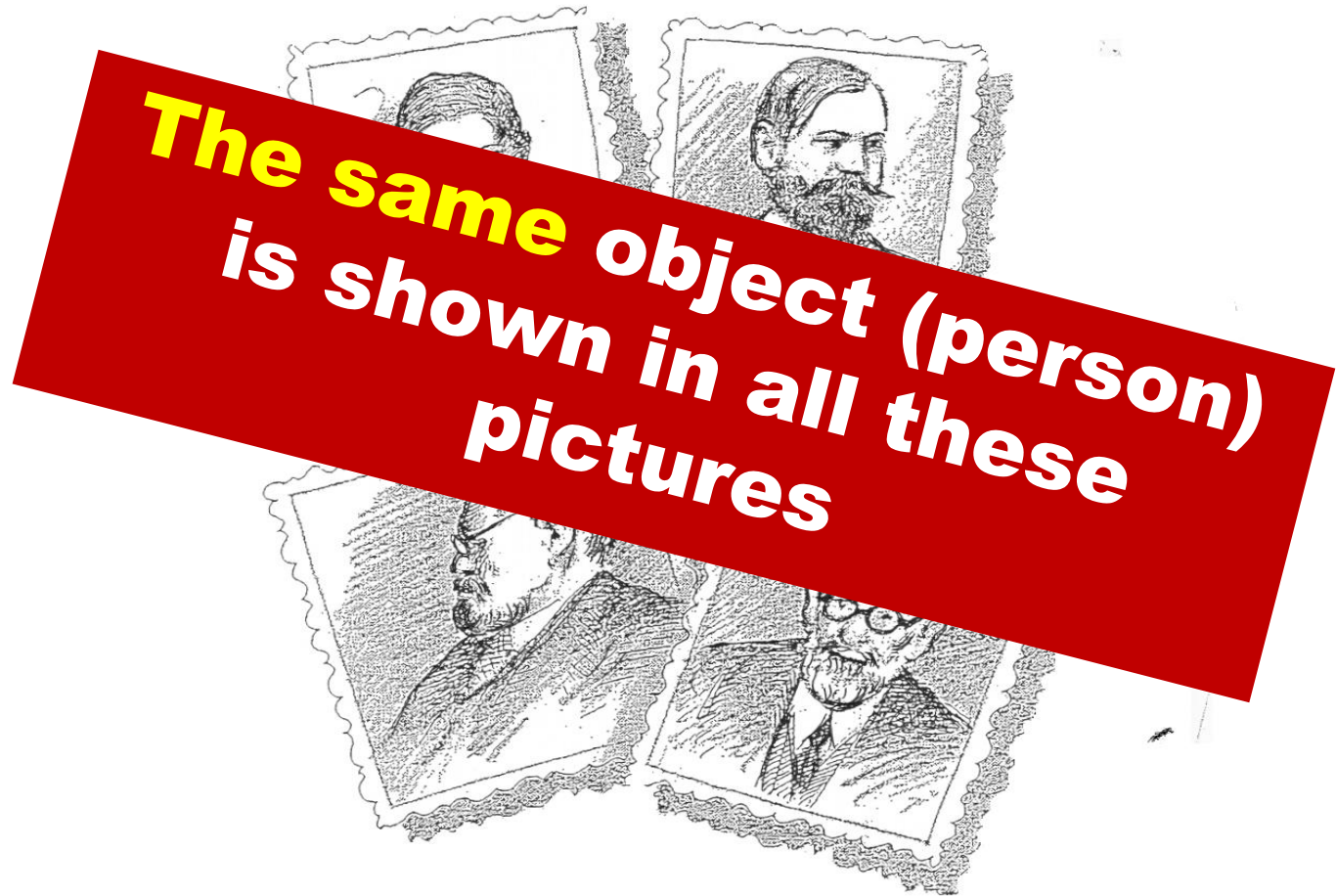
Another,
non-
dualist
take



**What makes us say:
“These pictures present the same person”?**



**What makes us say:
“These pictures present the same person”?**

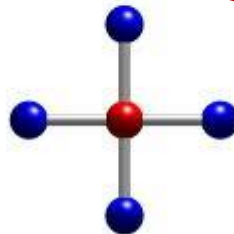


**What is
about**

**Yes! All of them
represent the same
mathematical object:
number 5**



**The same object is shown
in all these pictures?**





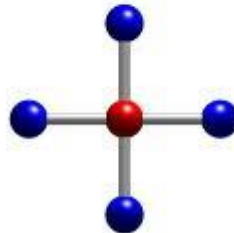
Henri Poincaré

Mathematics is
the art of giving
the same
name to
different
things

**What is “the same”
about these six things?**

When you

**The same PROCESS is
implicated in all these
pictures**



Conclusion: Number is but a **metaphor**

We speak about mathematical processes **as if** they were objects.

Conclusion:
Number is but a
metaphor

It is the
objectification
of the process of counting

Objectification

using a **sign** (word) as
if it signified
an **object** existing
in the world
independently of us

Objectification

Only after we adopt this objectified form of speech, we say that the number word/numerals **represent** the objects called numbers.

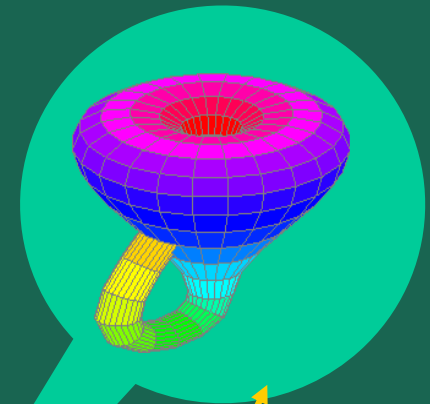
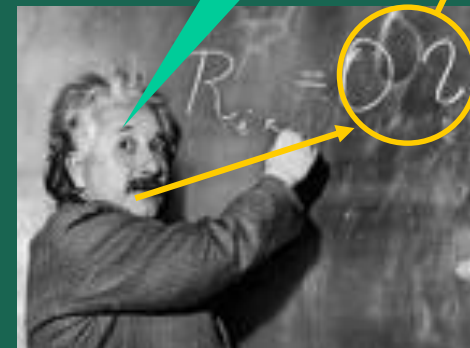
What is mathematical object?

Non-dualist take:

Discursive (commognitive) answer

**Mathematical
objects are
discursive
constructs:
they arise
through and for
the sake of
communication.**

8/6/2022





Carl Friedrich Gauss

Infinity is
merely a
*facon de
parler.*

Why talk in terms of objects?

What is this talking in
terms of mathematical
objects **good for?**

The birth of MO makes the mathematician's life easier



William
Thurston
1990

I remember as a child, in the fifth grade, coming to the amazing (to me) realization that the answer to **134 divided by 29** is **$134/29$** ...

What a **tremendous labor-saving device!** To me, “134 divided by 29” meant a certain **tedious chore**, while $134/29$ was **an object** with no implicit work.

Why talk in terms of objects?

What is this talking in terms of mathematical objects **good for?**

We can now say **more with less**

This form of talk changes **makes communication thriftier & more efficient**

Mathematical objects let you say more with less

$$(\sqrt{x})^3 = \sqrt{x^3}$$

If I extract the square root from x and raise the result to the third power, I get the same result as when I raise x to the 3rd power and extract the square root from it

The 3rd power of square root equals square root of the 3rd power

Mathematical objects let you say more with less

longer

$$(\sqrt{x})^3 = \sqrt{x^3}$$

shorter

If I extract a square root from x and raise the result to the third power, I get the same result as when I raise x to the 3rd power and extract square root from it

The 3rd power of square root equals square root of the 3rd power

Mathematical objects let you say more with less

longer

$$(\sqrt{x})^3 = \sqrt{x^3}$$

shorter

If I **extract** a square root from x and **raise** the result to the third power, I **get** the same result as when I **raise** x to the 3rd power and **extract** square root from it

The 3rd **power** of **square root** equals **square root of the 3rd power**

unobjectified

objectified

reification

replacing verb clauses
with a noun clauses

$$(\sqrt{x})^3 = \sqrt{x^3}$$

If I **extract** a square root from x and **raise** the result to the third power, I **get** the same result as when I **raise** x to the 3rd power and **extract** square root from it

The 3rd **power** of **square root** equals **square root of the 3rd power**

alienation

removing human subject

Objectification,
through the thriftiness, makes the
possibility of **extending** mathematics
practically **boundless**

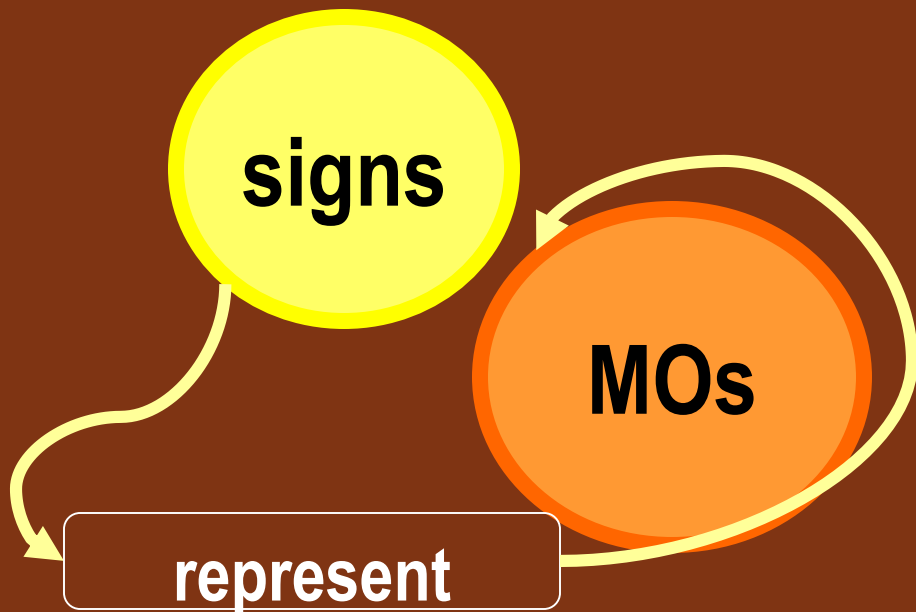
If I **extract** a square root from x and **raise** the result to the third power, I **get** the same result as when I **raise** x to the 3rd power and **extract** square root from it

The 3rd **power** of **square root** equals **square root of the 3rd power**

The relation between signs and MOs

Platonism

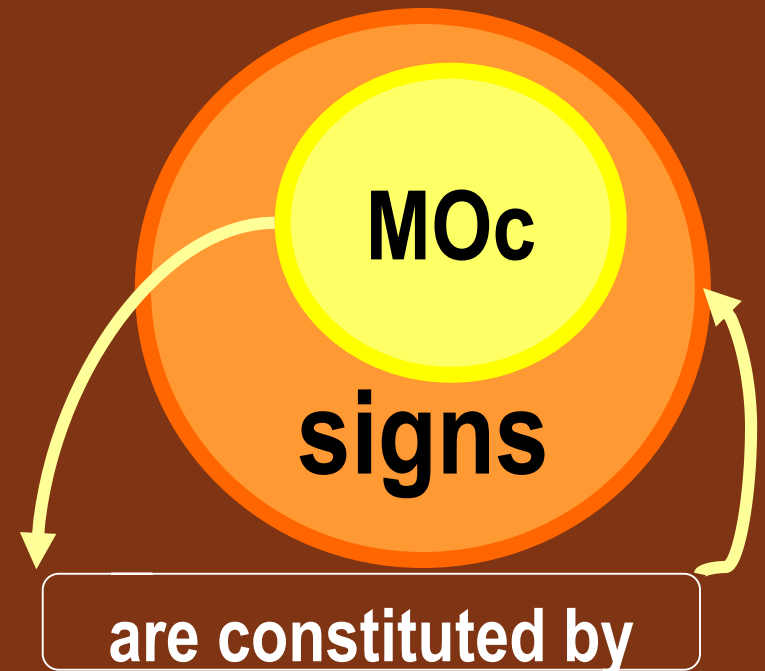
dualist



August 6, 2022

Discursivism (commognition)

monist



39

Plan of action

1. What is mathematical object?

From dualist to discursive view

2. What is mathematics?

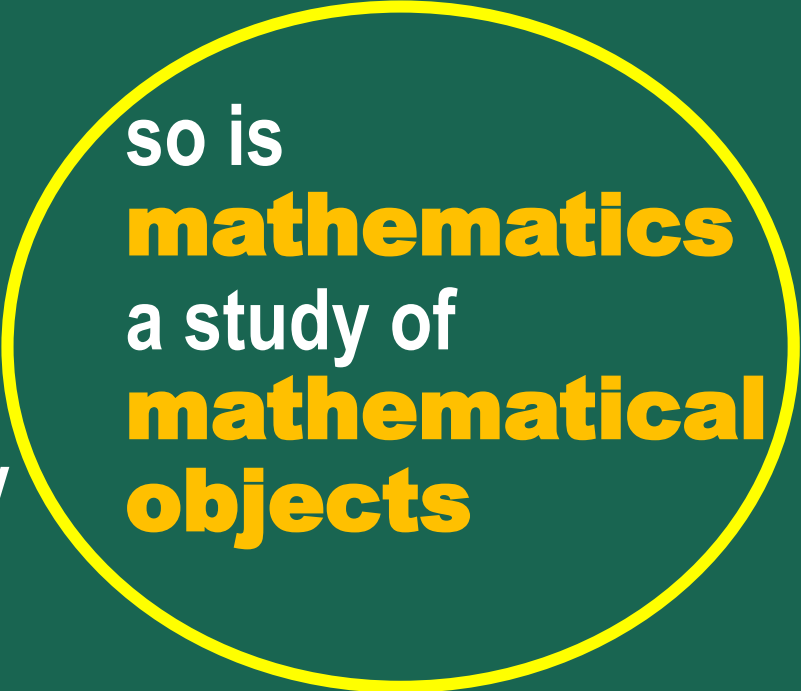
The discursive view

3. Why does it matter?

Educational implications

What is mathematics?

- Just like **biology** is a study of **living things** (plants, animals)
- and as **physics** is a study of **natural things** (moving bodies, light, etc.)



so is
mathematics
a study of
**mathematical
objects**

What is mathematics?

- Just like **biology** is a **study** of **living things** (plants, animals)

- and as **physics** is a **study** of **natural things** (moving bodies, light, etc.)

so is **mathematics** a **study** of **mathematical objects**

What does it mean **to study**
(mathematical) objects?

**To study a
certain type of
objects is to
tell potentially
useful (“true”)
stories about
them**

$S = \frac{1}{2}gt^2$
(Physics)

“The sum of angles in
a triangle equals 180° ”
(mathematics)

to study a
certain type of
objects is to
tell potentially
useful (“true”)
stories about
them

“Children initially
believe that
multiplication
makes bigger”
(research on
learning)

Story of X



noun

Story of X

consistent, cohesive, sequence of utterances that, when taken together, can be said to be **about X**

X is the **grammatical object** or **subject** of some of the utterances in the sequence.

Story of X

consistent, cohesive, sequence of utterances that, when taken together, can be said to be about X

There is **no contradiction** (logical conflict) between the utterances

Story of X

consistent, **cohesive**, sequence of utterances that, when taken together, can be said to be about X

The utterances are **inter-connected** chronologically, causally or logically.

to study a
certain type of
objects is to
tell potentially
useful ('true')
stories about
them

Humans are
storytellers

Stories are the best
tool we have for



**making
sense** of our
experience

8/6/2022

Humans are
storytellers

Stories are the best
tool we have for

**making
sense** of our
experience

deciding
how to **act**

Our practical actions are **mediated**
by stories about mathematical objects

A story counts as **useful** if it fulfills these two functions to our satisfaction

**making
sense** of our
experience

deciding
how to **act**

Scientific & mathematical stories
are told precisely for this purpose

**To be useful
("true"), these stories
have to be told**

**in special
ways, with
special tools**

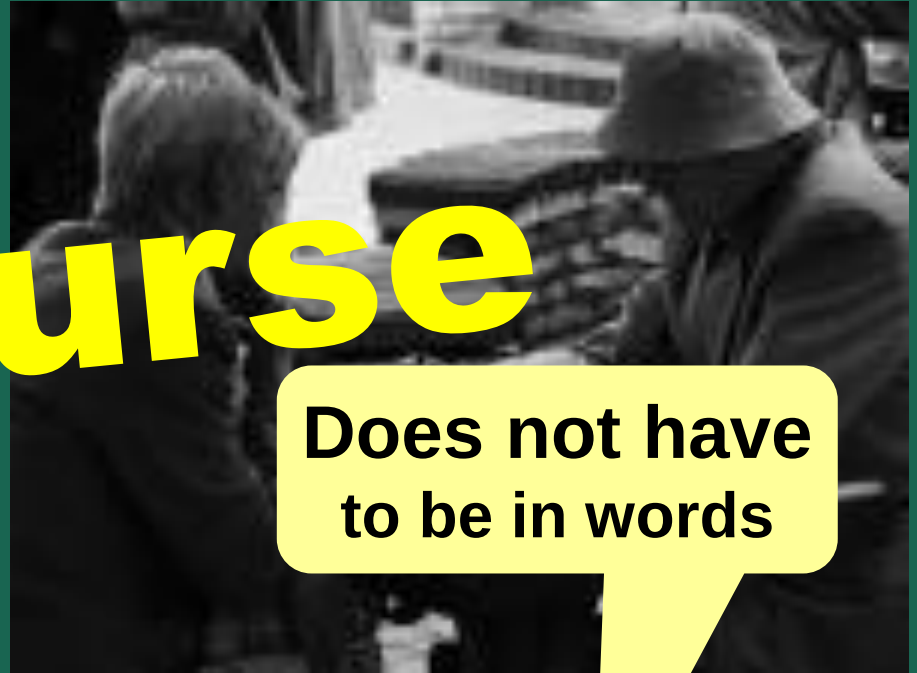
**= in a special
discourse**

Thinking!

discourse

May be with
others or
with oneself

Does not have
to be in words



→ a **form of communication**
[a communicational **game** that defines
a **community**]

Mathematics

Discourse

for telling stories of mathematical objects

Theory of MOs:

the ever growing set of
potentially useful
narratives of MOs

Plan of action

1. What is mathematical object?

From dualist to discursive view

2. What is mathematics?

The discursive view

3. Why does it matter?

Educational implications

**Some benefits
of conceptualizing
mathematics as discourse**

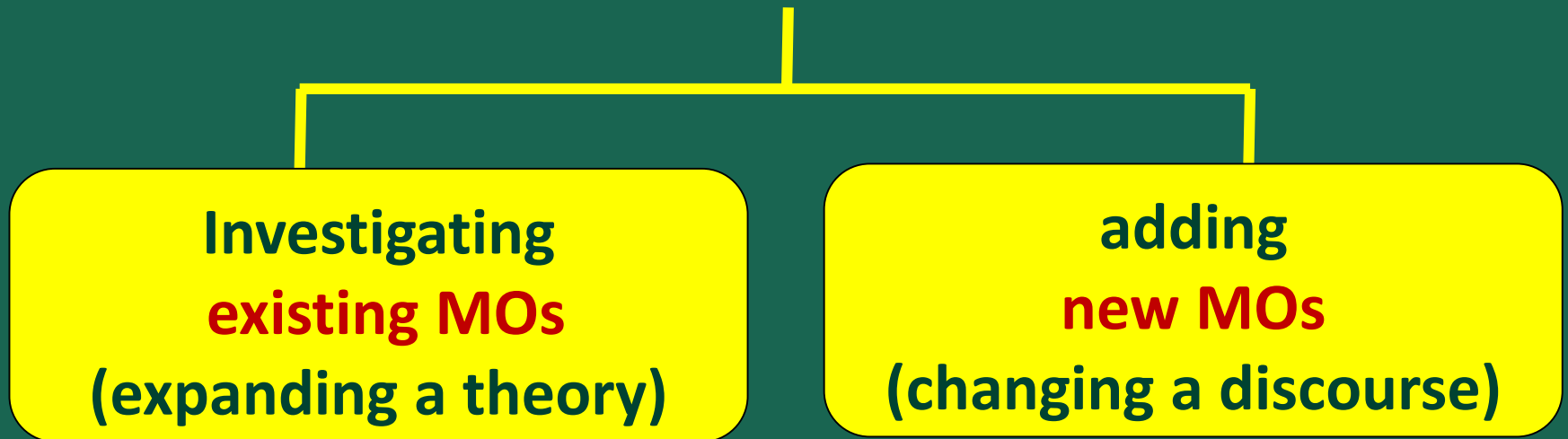
**A game-changing
insight on the
learning of mathematics**

learning mathematics – developing math. discourse

Investigating
existing MOs
(expanding a theory)

Example: negative integers have already been introduced, and now stories about them (theory of integers) must be built.

developing mathematics (whether historical or ontogenetic)



Example:
negative
numbers are
first introduced

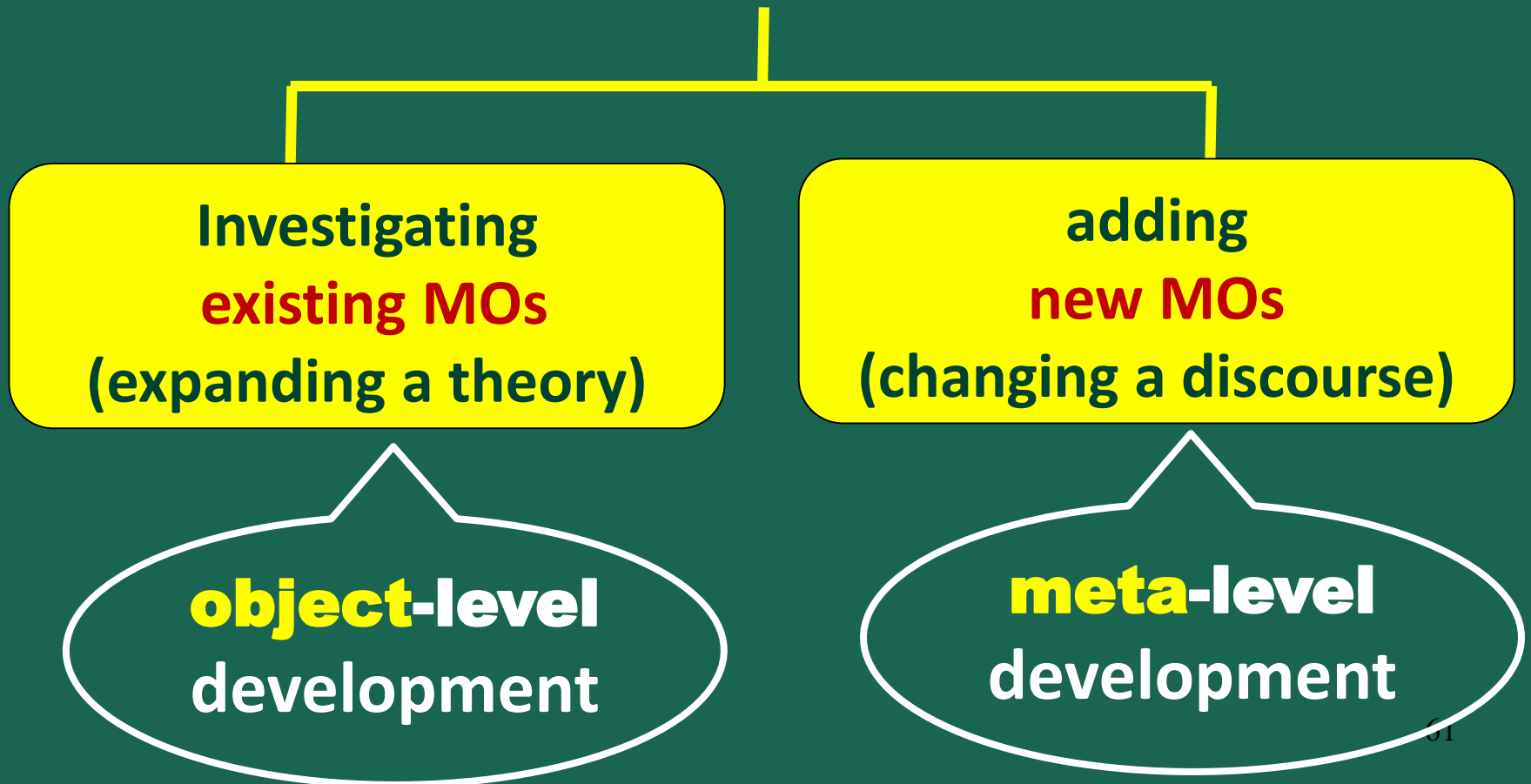
mathematics
cal or ontogenetic)

Investigating
existing MOs
(expanding a theory)

adding
new MOs
(changing a discourse)

-- transition from the discourse
on **unsigned numbers** to
the one on **signed numbers**

developing mathematics (whether historical or ontogenetic)



What's the difference in how the two levels of learning may occur?

object-level

learning can
happen

The discourse change
is a matter of

**logically
necessary**

What's the difference in how the two levels of learning may occur?

object-level

learning can
happen

The discourse change
is a matter of

**logically
necessary**

meta-level learning

Not that simple!

The required meta-
discursive change, the
matter of

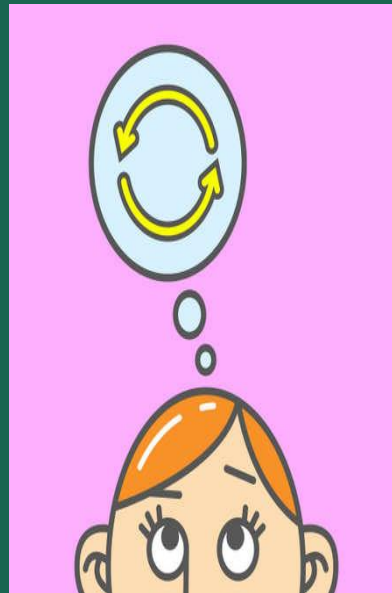
contingency

Circularity

of requirements for learning

To construct
a new
object, the
learner has
to
participate
in the
discourse
about it

but



in order to
participate in
a that
discourse, she
has to **have**
already
constructed
the new
object



**Mathematics teacher
as seen through his students' eyes**

Circularity

of requirements for learning

To construct a concept in order to
be able to use it in

**How to
overcome
the circularity?**

discuss how
about it **object**

Meta-level learning – what can the student do?

imitating
expert participant

constantly
reflecting
on the experts' reasons

reflective imitation

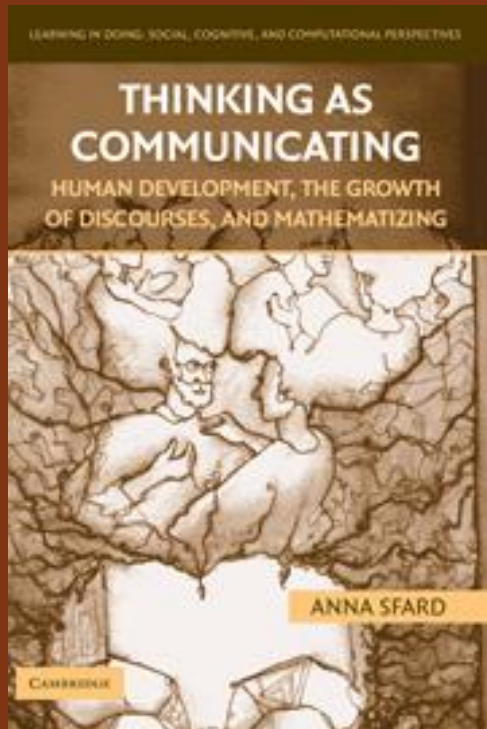
Meta-level learning – what can the student do?

imitating
expert participant

constantly
reflecting
on the experts' reasons

reflective imitation

To learn more about the discursive vision of mathematics and its ramification go to



ICMI AMOR
ICMI Awardees Multimedia
Online Resources

<https://www.mathunion.org/icmi/awards/amor/anna-sfard-unit>