Motivic cohomology: past, present and future

Marc Levine
Universität Duisburg-Essen

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The Beilinson-Lichtenbaum conjectures

Sources for the Beilinson-Lichtenbaum conjectures

- Quillen's algebraic K-theory (1972)
- Conjectures (Birch and Swinnerton-Dyer, Birch and Tate, Lichtenbaum, Deligne, Bloch, ...) about orders of vanishing and leading terms of zeta-functions and L-functions.
- Bloch-Ogus twisted duality theory (1974); axioms extended by Gillet (1981).
- The Quillen-Lichtenbaum conjectures: $K_n(X, \mathbb{Z}/m) \cong K_n^{\text{ét}}(X, \mathbb{Z}/m)$ for $n >> 0$. (1975).

K-theory and zeta-values

F: number field, $\mathcal{O}_F$: ring of integers

Quillen: $K_n(\mathcal{O}_F)$ is a finitely generated abelian group.

Borel: $K_{2n}(\mathcal{O}_F)$ is torsion for $n \geq 1$, and $K_{2n+1}(\mathcal{O}_F)$ has rank

$$\operatorname{rank} K_{2n+1}(\mathcal{O}_F) = \operatorname{ord}_{s=-n} \zeta_F(s)$$

Lichtenbaum's conjecture: $\zeta_F(-n)^*$ should be given by

$$\frac{|K_{2n}(\mathcal{O}_F)_{\text{tor}}| \cdot \operatorname{Reg}(K_{2n+1}(\mathcal{O}_F)/\text{tor})}{|K_{2n+1}(\mathcal{O}_F)_{\text{tor}}|}$$

Correct for $n = 0$ (class number formula + functional equation),

Fails for $n = 1$, $F = \mathbb{Q}$: $K_3^M(\mathbb{Z}) = \mathbb{Z}/2 \subset K_3(\mathbb{Z}) = \mathbb{Z}/48$.

Bloch-Ogus theories

A Bloch-Ogus theory is a bi-graded cohomology theory

$$X \mapsto H^p(X,q), \;\; q = \text{"Tate twist"}$$

with 1st Chern class homomorphism $+ \ldots$.

(some) zeta values $\leftrightarrow$ étale cohomology (Tate, Coates, Wiles, Lichtenbaum, ...).

Étale cohomology **is** a Bloch-Ogus theory: $H^p(X,q) := H^p_{\text{ét}}(X, \mu_m^{\otimes q})$, but K-theory is **not**.

Example

$K_3(\mathbb{Z})$ is detected in étale cohomology groups with two different twists:

$$H^3_{\text{ét}}(\mathbb{Z}[1/6], \mathbb{Z}_2(3)) = \mathbb{Z}/2 \longleftrightarrow K_3^M(\mathbb{Z}) \subset K_3(\mathbb{Z}) = \mathbb{Z}/48$$

and the quotient $\mathbb{Z}/24$ is $H^1_{\text{ét}}(\mathbb{Z}[1/6], \mathbb{Z}_2(2) \oplus \mathbb{Z}_3(2))$

Is there a Bloch-Ogus theory approximating K-theory, that is a better fit to the zeta values?

Adams-graded K-theory

Beilinson (1983):

$$X \mapsto H^p(X, \mathbb{Q}(q)) := k^q\text{-eigenspace for } \psi_k \text{ on } K_{2q-p}(X)_{\mathbb{Q}}$$

is the universal Bloch-Ogus theory with $\mathbb{Q}$-coefficients.

Perhaps the universal *integral* Bloch-Ogus theory is the answer?

The Beilinson-Lichtenbaum conjectures

(1983) Beilinson and Lichtenbaum independently conjecture:

There is a presheaf of complexes on $\mathbf{Sm}_k$, $X \mapsto \Gamma_X(q)$, defining the universal Bloch-Ogus theory by

$$H^p(X, \mathbb{Z}(q)) := \mathbb{H}^p(X_{\mathrm{Zar}}, \Gamma_X(q))$$

The $\Gamma_X(q)$ should have a close relation with algebraic K-theory, Milnor K-theory, étale cohomology and, in positive characteristic, with log differentials (added by Milne).

The Beilinson-Lichtenbaum conjectures

(1984) Beilinson conjectures:

There is an abelian tensor category of "mixed motivic sheaves on X", $\mathcal{MM}_X$, such that

- $X \mapsto D^b(\mathcal{MM}_X)$ admits a Grothendieck six-functor formalism:

$$Lf^* \dashv Rf_*, \ \ f_! \dashv f^!, - \otimes -, \mathcal{H}om(-,-)$$

- One has Tate sheaves $\mathbb{Z}_X(q) \in \mathcal{MM}_X$, with $\Gamma_X(q) = R\pi_{X*}(\mathbb{Z}_X(q))$.

Thus

$$H^p(X, \mathbb{Z}(q)) = \mathrm{Ext}^p_{\mathcal{MM}_X}(\mathbb{Z}_X(0), \mathbb{Z}_X(q)),$$

analogous to

$$H^p_{sing}(T, \mathbb{Z}) = \mathrm{Ext}^p_{Sh_T^{\mathbf{Ab}}}(\mathbb{Z}_T, \mathbb{Z}_T).$$

Hence:

$$\boxed{H^p(X, \mathbb{Z}(q)) \text{ is } \textit{motivic cohomology}}$$

Bloch's higher Chow groups and Suslin's algebraic homology

The algebraic *n*-simplex

The topological *n*-simplex is

$$\Delta^n_{top} = \{\sum_{i=0}^{n} t_i = 1, t_i \geq 0\} \subset \mathbb{R}^{n+1},$$

They form a cosimplicial space $n \mapsto \Delta^n_{top}$ via the face and degeneracy maps.

The *algebraic n-simplex* is:

$$\Delta^n := \{\sum_{i=0}^{n} t_i = 1\} \subset \mathbb{A}^{n+1}.$$

The face and degeneracy maps for Δ^n_{top} are linear in the barycentric coordinates $t_0, \ldots, t_n$, defining the cosimplicial scheme

$$\Delta^* := \ n \mapsto \Delta^n$$

For a k-scheme X, we have the cosimplicial scheme $\Delta^* \times X$.

Bloch's cycle complex

Bloch: let

$$z^q(X,n) := \mathbb{Z}\{Z \subset \Delta^n \times X \text{ a codimension } q \text{ subvariety} + \dots\}$$

The $+\dots$ are conditions of "good intersection with faces" $\rightsquigarrow$
For each cosimplicial structure map $g : \Delta^m \times X \to \Delta^n \times X$, the pullback

$$g^* : z^q(X,n) \to z^q(X,m)$$

is defined.

The pullback by faces $\partial_i : \Delta^{n-1} \times X \to \Delta^n \times X$ defines the differential

$$d := \sum_i (-1)^i \partial_i^* : z^q(X,n) \to z^q(X,n-1)$$

giving rise to

$$\boxed{(z^q(X,*),d) =: \text{ Bloch's } \textit{cycle complex}.}$$

Bloch's higher Chow groups

Definition (Bloch 1986)

Bloch's *higher Chow groups of X* are defined by

$$\mathrm{CH}^q(X, n) := H_n(z^q(X, *))$$

$\mathrm{CH}^q(X, n)$ is *algebraic Borel-Moore homology*

The corresponding Beilinson-Lichtenbaum complexes would be:

$$\Gamma_X^{Bl}(q)^* = \text{sheafification of: } U \underset{\text{open}}{\subset} X \mapsto z^q(U, 2q-*),$$

Motivic cohomology would be:

$$H^p(X, \mathbb{Z}(q)) := \mathbb{H}^p(X_{\mathrm{Zar}}, \Gamma_X^{Bl}(q)) = H_{2q-p}(z^q(X,*)) = \mathrm{CH}^q(X, 2q-p)$$

Examples

- $\mathrm{CH}^q(X,0) = \mathrm{CH}^q(X)$, the classical Chow group.
- For a field F, $\mathrm{CH}^q(\mathrm{Spec}\, F, m) = 0$ for $m < q$ and

(Totaro, Nesterenko-Suslin 1992)

$$\boxed{\mathrm{CH}^q(\mathrm{Spec}\, F, q) = K_q^M(F)}$$

where $K_*^M(F) := (F^\times)^{\otimes *}/\langle a \otimes (1-a), a \neq 0, 1\rangle$ is Milnor K-theory.

Suslin homology

(1986-Luminy) Suslin defines *algebraic singular homology*.

Replace singular simplices $\sigma : \Delta^n_{top} \to T$ with "algebraic multi-valued maps" $\Delta^n \to X$:

$$C_n^{Sus}(X, \mathbb{Z}) := \mathbb{Z}\{Z \subset \Delta^n \times X \text{ with } Z \to \Delta^n \text{ finite and surjective.}\}$$

Restriction to faces $\partial_i : \Delta^{n-1} \to \Delta^n$ defines a differential

$$\partial := \sum_i (-1)^i \partial_i^* : C_n^{Sus}(X, \mathbb{Z}) \to C_{n-1}^{Sus}(X, \mathbb{Z})$$

giving *Suslin's algebraic homology complex* $C_*^{Sus}(X, \mathbb{Z})$.

Definition (Suslin 1986)

The *Suslin homology* of X is

$$H_n^{Sus}(X, \mathbb{Z}) := H_n(C_*^{Sus}(X, \mathbb{Z}))$$

Motivic categories

Categorical motivic cohomology

Voevodsky: Suslin's idea of multi-valued algebraic maps extends to yield the category Cor_k of *finite correspondences*.

$$\mathrm{Cor}_k(X, Y) = \mathbb{Z}\{Z \subset X \times Y \text{ subvariety} \mid Z \to X \text{ finite and surjective.}\}$$
$$\text{“} = \text{''}\{\text{multivalued algebraic maps } X \to Y\}^{gp}.$$

Definition

Voevodsky's triangulated category of effective motives over k is:

$$\mathrm{DM}^{eff}(k) := L_{\mathbb{A}^1,\mathrm{Nis}_{q.i.}}(\text{additive } C(\mathbf{Ab})\text{-presheaves on } \mathrm{Cor}_k)$$

Here $L_{\mathbb{A}^1,\mathrm{Nis}_{q.i.}}$:= localization to invert

1. maps of presheaves that are quasi-isomorphisms on the Nisnevich stalks, and
2. the maps of representable presheaves $\mathrm{Cor}_k(-, X \times \mathbb{A}^1) \xrightarrow{p_1} \mathrm{Cor}_k(-, X)$.

That is: $\mathbb{A}^1$ is contracted to a point.

Categorical motivic cohomology

Define: $M(X) := L_{\mathbb{A}^1,Nis}(\mathrm{Cor}_k(-,X))$: the *motive* of $X \in \mathbf{Sm}_k$
$\mathbb{Z}(1) := M(\mathbb{P}^1)/M(\infty)[-2]$, $\mathbb{Z}(q) := \mathbb{Z}(1)^{\otimes q}$: Tate motives

Invert $- \otimes \mathbb{Z}(1)$ to enable duality:

$$\mathrm{DM}(k) := \mathrm{DM}^{eff}(k)[(- \otimes \mathbb{Z}(1))^{-1}]$$

Define categorical *motivic cohomology* and *motivic homology* by

$$H^p(X,\mathbb{Z}(q)) := \mathrm{Hom}_{\mathrm{DM}(k)}(M(X),\mathbb{Z}(q)[p]),$$
$$H_n(X,\mathbb{Z}) := \mathrm{Hom}_{\mathrm{DM}(k)}(\mathbb{Z}(0)[n],M(X)).$$

Theorem (Voevodsky-Suslin-Friedlander (2000))

Bloch's higher Chow groups compute motivic cohomology, and Suslin homology computes motivic homology:
$H^p(X,\mathbb{Z}(q)) \cong \mathrm{CH}^q(X,2q-p)$, $H_n(X,\mathbb{Z}) \cong H_n^{Sus}(X,\mathbb{Z})$.

Motivic complexes

$\mathbb{Z}(q) \in \mathrm{DM}^{\mathrm{eff}}(k)$ can be viewed as a presheaf $\underline{\mathbb{Z}}(q)$ of complexes on $\mathbf{Sm}_k$, satisfying most of the Beilinson-Lichtenbaum axioms:

1. $\mathbb{H}^p(X_{\mathrm{Zar}}, \underline{\mathbb{Z}}(q)) = \mathbb{H}^p(X_{\mathrm{Nis}}, \underline{\mathbb{Z}}(q)) = H^p(X, \mathbb{Z}(q))$.

2. $\mathcal{H}^m(\underline{\mathbb{Z}}(q)) = 0$ for $m > q$, $\mathcal{H}^q(\underline{\mathbb{Z}}(q)) = \mathcal{K}^M_q$

3. (Bloch-Lichtenbaum, Friedlander-Suslin) There is a filtration on $K(X)$ with layers $\underline{\mathbb{Z}}(q)(X)[2q]$, giving the motivic Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

4. (Bloch-Kato/Beilinson-Lichtenbaum conjectures, proved by Voevodsky, Rost ...)

$$\underline{\mathbb{Z}}(q)/m \cong \tau_{\leq q} R\Gamma_{\text{ét}}(-, \mu_m^{\otimes q}) \text{ for } (m, \mathrm{char}\, k) = 1$$

5. (Geisser-L.) For $\mathrm{char}\, k = p > 0$,

$$\underline{\mathbb{Z}}(q)/p \cong \Omega^q_{\log}[-q],\ \underline{\mathbb{Z}}(q)/p^m \cong \log W_m\Omega^q[-q].$$

Motivic complexes

⚠ One Beilinson-Lichtenbaum axiom is missing.

Beilinson-Soulé vanishing: $\mathcal{H}^m(\underline{\mathbb{Z}}(q)) = 0$ for $m \leq 0$, $q > 0$, is still not known for $q \geq 2$!

On the plus side: the remaining properties yield the Quillen-Lichtenbaum conjectures:

$$K_n(X, \mathbb{Z}/\ell^\nu) \cong K_n^{\text{ét}}(X, \mathbb{Z}/\ell^\nu)$$

for $n \geq cd_\ell(X) - 1$.

Motivic homotopy theory

Homotopy theory is relies on: the category of spaces **Spc**, the unstable homotopy category $\mathcal{H} = L_{W.E.}\mathbf{Spc}$ and the stable homotopy category of spectra $\mathrm{SH} = \mathcal{H}_\bullet[\Sigma_{S^1}^{-1}]$.

Morel and Voevodsky define motivic versions of these.

Definition (Morel-Voevodsky, Voevodsky)

$\mathbf{Spc}(k)$:=presheaves of spaces on $\mathbf{Sm}_k$
$\mathcal{H}(k) := L_{\mathbb{A}^1, Nis_{W.E.}}(\mathbf{Spc}(k))$: the motivic unstable homotopy category
$\mathrm{SH}(k) := \mathcal{H}_\bullet(k)[\Sigma_{\mathbb{P}^1}^{-1}]$: the motivic stable homotopy category of $\mathbb{P}^1$-spectra.

Inverting $\Sigma_{\mathbb{P}^1}$ is needed for motivic Spanier-Whitehead-Voevodsky duality in $\mathrm{SH}(k)$.

Motivic homotopy theory

- Replacing the 1-parameter family of invertible suspension operators $\{\Sigma^n := \Sigma^n_{S^1}\}$ on SH is the 2-parameter family $\{\Sigma^{a,b} := \Sigma^{a-b}_{S^1}\Sigma^b_{\mathbb{G}_m}\}$ on $\mathrm{SH}(k)$.
- For $X \in \mathbf{Sm}_k$, we have the infinite $\mathbb{P}^1$-suspension spectrum $\Sigma^\infty_{\mathbb{P}^1} X_+ \in \mathrm{SH}(k)$.
- $\mathrm{SH}(k)$ is a tensor triangulated category, with unit $\mathbb{S}_k := \Sigma^\infty_{\mathbb{P}^1}\mathrm{Spec}\, k_+$.
- A $\mathbb{P}^1$-spectrum $E \in \mathrm{SH}(k)$ defines a bi-graded cohomology theory on $\mathbf{Sm}_k$ by:
$$E^{p,q}(X) := [\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^{p,q} E]_{\mathrm{SH}(k)}.$$
- E has bi-graded homotopy sheaves $\pi_{p,q}E$: just sheafify the presheaf
$$X \in \mathbf{Sm}_k \mapsto [\Sigma^{p,q}\Sigma^\infty_{\mathbb{P}^1} X_+, E]_{\mathrm{SH}(k)} = E^{-p,-q}(X).$$

The motivic cohomology spectrum

Voevodsky constructed a $\mathbb{P}^1$-spectrum $H\mathbb{Z}_{Voev}$ in $\mathrm{SH}(k)$ representing motivic cohomology and a $\mathbb{P}^1$-spectrum KGL representing Quillen K-theory:

For $X \in \mathbf{Sm}_k$

$$H\mathbb{Z}^{p,q}_{Voev}(X) = H^p(X, \mathbb{Z}(q)),$$

$$\mathrm{KGL}^{p,q}(X) = K_{2q-p}(X).$$

Voevodsky's slice tower

The classical Moore-Postnikov tower in SH filters a spectrum E by connectivity,

$$\ldots \to \tau_{\geq n+1}E \to \tau_{\geq n}E \to \ldots \to E,$$

with $\tau_{\geq n}E$ the $(n-1)$-connected cover of E.

Voevodsky defined a filtration on $\mathrm{SH}(k)$ by "$\mathbb{P}^1$-connectivity". This yields his *slice tower*,

$$\ldots \to f_{n+1}E \to f_nE \to \ldots \to E,$$

with f_nE the $\mathbb{P}^1$-$(n-1)$-connected cover of E.

The layer $s_nE := f_nE/f_{n+1}E$ is called the nth slice of E.

Voevodsky's slice tower

Theorem (Voevodsky, L., Bachmann-Elmanto)

$s_0(\mathbb{S}_k) = H\mathbb{Z}_{Voev}$, $s_n\mathrm{KGL} = \Sigma^n_{\mathbb{P}^1} H\mathbb{Z}_{Voev}$

This corresponds to: $\pi_0^s(\mathbb{S}) = \mathbb{Z}$, and $\pi_{2n}^s(\mathrm{KU}) = \mathbb{Z}$, $\pi_{2n+1}^s(\mathrm{KU}) = 0$.

The spectral sequence arising from the slice tower for KGL recovers the Bloch-Lichtenbaum/Friedlander-Suslin spectral sequence, and corresponds to the Atiyah-Hirzebruch spectral sequence for KU.

Modules over motivic cohomology

Theorem (Röndigs-Østvær, Kelly)

$H\mathbb{Z}_{Voev}$ refines to an E_∞ ring spectrum $H\mathbb{Z}$, and $\mathrm{DM}(k) \cong H\mathbb{Z} - \mathbf{Mod}$ (invert p if $\operatorname{char} k = p > 0$).

Universality of motivic cohomology

Theorem (Hopkins-Morel, Hoyois)

Motivic cohomology is the universal Bloch-Ogus theory (oriented cohomology theory with additive formal group law) on $\mathbf{Sm}_k$ *(after inverting* p *if* $\mathrm{char}\, k = p > 0$*).*

The proof is highly homotopical, relying on:

- Voevodsky's algebraic cobordism spectrum MGL
- Universality of MGL (Panin-Pimenev-Röndigs, Vezzosi)
- Motivic Landweber exactness for MGL (Naumann, Spitzweck, Østvær)
- Voevodsky's motivic Steenrod algebra

Motivic sheaves: Motivic categories over a general base

The motivic categories over a general base

Voevodsky-Morel gave the construction of $\mathcal{H}(k)$, $\mathrm{SH}(k)$ over a "general" base-scheme B, with the same definitions:

$$\mathbf{Spc}(B) = \text{presheaves of spaces on } \mathbf{Sm}_B$$

$$\mathcal{H}(B) = L_{\mathbb{A}^1, Nis}(\mathbf{Spc}(B))$$

$$\mathrm{SH}(B) = \mathcal{H}_\bullet(B)[\Sigma_{\mathbb{P}^1}^{-1}]$$

The six-functor formalism for $B \mapsto \mathrm{SH}(B)$ was established by Ayoub, Cisinski-Déglise, Hoyois.

Voevodsky's slice tower and his algebraic K-theory spectrum KGL extend directly. However: over a general B, $\mathrm{KGL}_B \in \mathrm{SH}(B)$ represents Weibel's K-theory made $\mathbb{A}^1$-homotopy invariant, KH, via

$$\mathrm{KGL}_B^{p,q}(X) = \mathrm{KH}_{2q-p}(X), \; X \in \mathbf{Sm}_B,$$

(not Quillen K-theory, unless B is regular).

Spitzweck's motivic cohomology spectrum

The construction of a suitable $H\mathbb{Z}_B$ is more complicated. Here is Spitzweck's method in rough outline.

1. Start with a naive extension of Bloch's complexes $X \mapsto z^q(X, 2q-*)$ to a presheaf $\mathbb{Z}^{BL}(q)$ on $\mathbf{Sm}_A$, A a Dedekind domain. This gives a reasonable additive theory, but has no ring structure.

2. We restrict to $A = \mathbb{Z}$. The version with $\mathbb{Q}$-coefficients defines an E_∞ $\mathbb{P}^1$-spectrum $H\mathbb{Q} \in \mathrm{SH}(\mathbb{Z})$, as a summand of $\mathrm{KGL} \otimes \mathbb{Q}$ (Cisinski-Déglise).

3. Geisser: there is a distinguished triangle

$$\mathbb{Z}^{BL}(q)/p^n \to \tau_{\leq q} Rj_* R\epsilon_* \mu_{p^n}^{\otimes q} \to i_* \log W_n\Omega^{q-1}[-q+1]$$

with $\epsilon : \mathbf{Sm}_{\mathbb{Z}[1/p],\mathrm{Nis}} \to \mathbf{Sm}_{\mathbb{Z}[1/p],\text{ét}}$ the morphism of sites and i, j: $\mathrm{Spec}\, \mathbb{Z}[1/p] \xrightarrow{j} \mathrm{Spec}\, \mathbb{Z} \xleftarrow{i} \mathrm{Spec}\, \mathbb{F}_p$.

Spitzweck gluing

4. Spitzweck: use Geisser's distinguished triangle to fit the p-completions $\mathbb{Z}^{BL}(q)^{\wedge_p}$ into an E_∞ $\mathbb{P}^1$-spectrum D_p. Let $D := \prod_p D_p$.

Define the E_∞ $\mathbb{P}^1$-spectrum $H\mathbb{Z} \in \mathrm{SH}(\mathbb{Z})$ by gluing in the category $\{E_\infty\text{-}\mathbb{P}^1\text{-spectra}\}$:

$$\begin{array}{ccc} H\mathbb{Z} & \longrightarrow & D \\ \downarrow & & \downarrow \\ H\mathbb{Q} & \longrightarrow & D_{\mathbb{Q}} \end{array}$$

5. For $\pi : B \to \operatorname{Spec}\mathbb{Z}$, define $H\mathbb{Z}_B := \pi^* H\mathbb{Z} \in \mathrm{SH}(B)$.

Theorem (Spitzweck)

Let A be a Dedekind domain, $B := \operatorname{Spec} A$ and let X be a smooth, separated, finite-type scheme over B. Then

$$H\mathbb{Z}_B^{p,q}(X) \cong \mathrm{CH}^q(X, 2q-p) := \mathbb{H}^p(X_{\mathrm{Nis}}, \mathbb{Z}^{BL}(q)_{|X}).$$

Derived mixed motivic sheaves

Definition (The ∞-category of motives over a base)

$$\boxed{\mathrm{DM}(B) := H\mathbb{Z}_B - \mathbf{Mod}}$$

By the theorem of Röndigs-Østvær/Kelly, this definition of $\mathrm{DM}(B)$ agrees with Voevodsky's if $B = \operatorname{Spec} k$ (after inverting p if $\operatorname{char} k = p > 0$).

$B \mapsto H\mathbb{Z}_B - \mathbf{Mod}$ inherits a six-functor formalism from $\mathrm{SH}(-)$, giving an ∞-derived version of Beilinson's (still conjectural!) abelian category of mixed motivic sheaves for general schemes.

Motivic cohomology and homotopy invariant K-theory

The main results on slices of $\mathbb{S}_k$ and KGL_k extend to arbitrary base-schemes.

Theorem

1. *(Bachmann-Elmanto-Morrow)* $s_0\mathbb{S}_B = H\mathbb{Z}_B$.
2. *(Bachmann)* $s_n\mathrm{KGL}_B = \Sigma^n_{\mathbb{P}^1} H\mathbb{Z}_B$

The Atiyah-Hirzebruch spectral sequence for a general scheme B,

$$E_2^{p,q}(B) = H\mathbb{Z}_B^{p-q,-q}(B) \Rightarrow \mathrm{KH}_{-p-q}(B),$$

is a consequence.

Beyond motivic cohomology

• Motivic cohomology as represented by $H\mathbb{Z}_B \in \mathrm{SH}(B)$ is $\mathbb{A}^1$-invariant.

Many interesting invariants are *not* $\mathbb{A}^1$-invariant:

- K-theory of singular schemes
- cristalline cohomology, Hochschild homology
- topological cyclic homology

Recent extensions of motivic cohomology attempt to capture this type of information.

• Motivic cohomology gives information on algebraic geometry over $\mathbb{C}$, but only mod 2 information on the cohomology of the real points of varieties over $\mathbb{R}$. Extending motivic cohomology to involve quadratic forms gives better information about real algebraic varieties and new arithmetic information over general fields.

Non-$\mathbb{A}^1$-invariant motivic cohomology

The cyclotomic trace map

Elmanto-Morrow construct non-$\mathbb{A}^1$-invariant motivic complexes $\mathbb{Z}(q)^{\mathrm{mot}}$ for an arbitrary scheme over a field.

Their construction relies on a homotopy cartesian square and an identity:

$$\begin{array}{ccccc} & & K(X) & \xrightarrow{\mathrm{Tr}} & \mathrm{TC}(X) \\ & & \downarrow & & \downarrow \\ KH(X) & = & K_{\mathrm{cdh}}(X) & \xrightarrow{\mathrm{Tr}_{\mathrm{cdh}}} & \mathrm{TC}_{\mathrm{cdh}}(X) \end{array}$$

The homotopy cartesian square is the result of a long development in the study of the cyclotomic trace and excision for K-theory: Bökstedt-Hsiang-Madsen, Suslin-Wodzicki, Cortiñas, Geisser-Hesselholt, culminating in the work of Land-Tamme.

The identity $KH(X) = K_{\mathrm{cdh}}(X)$ is due to Haesemeyer in characteristic zero, and in general to Kerz-Strunk-Tamme, relying on Land-Tamme, Kelly-Morrow.

Non-$\mathbb{A}^1$-invariant motivic complexes

In positive characteristic, Elmanto-Morrow construct motivic complexes $\mathbb{Z}(q)^{\mathrm{mot}}$ as follows:

Step 1. Extend the Bloch-Voevodsky complexes $\underline{\mathbb{Z}}(q)$ on $\mathbf{Sm}_k$ to $\tilde{\mathbb{Z}}(q)$ on $\mathbf{Sch}_k$ by left Kan extension, then cdh-localize: $\mathbb{Z}(q)^{\mathrm{cdh}} := L_{\mathrm{cdh}}\tilde{\mathbb{Z}}(q)$.

Step 2. Bhatt-Morrow-Scholze define a "Postnikov" filtration on $\mathrm{TC}(X)$, giving the layers $\mathbb{Z}_p(q)^{\mathrm{syn}}(X)[2q]$, related to prismatic cohomology.

Bachmann-Elmanto-Morrow: $K_{\mathrm{cdh}}(X) = \mathrm{KH}(X)$ has its induced slice filtration with layers $\mathbb{Z}(q)^{\mathrm{cdh}}(X)[2q]$.

Step 3. Elmanto-Morrow show that the BMS filtration and the slice filtration agree after mapping to $\mathrm{TC}_{\mathrm{cdh}}(X)$.

Step 4.

$$\begin{array}{ccc}
 & & \mathrm{TC}(X) \\
 & & \downarrow \\
KH(X) = K_{\mathrm{cdh}}(X) & \to & \mathrm{TC}_{\mathrm{cdh}}(X)
\end{array}
\quad \overset{\text{Take layers and glue}}{\rightsquigarrow} \quad
\begin{array}{ccc}
\mathbb{Z}(q)^{\mathrm{mot}} & \longrightarrow & \mathbb{Z}_p(q)^{\mathrm{syn}} \\
\downarrow & & \downarrow \\
\mathbb{Z}(q)^{\mathrm{cdh}} & \to & L_{\mathrm{cdh}}\mathbb{Z}_p(q)^{\mathrm{syn}}
\end{array}$$

There is a similar construction in characteristic zero

Non-$\mathbb{A}^1$-invariant motivic cohomology

Some properties of $\mathbb{Z}(q)^{\mathrm{mot}}$:

1. $\mathbb{Z}(q)^{\mathrm{mot}} = \mathbb{Z}(q)$ on $\mathbf{Sm}_k$
2. (Bachmann-Elmanto-Morrow) $\mathbb{Z}(q)^{\mathrm{cdh}}$ represents $H\mathbb{Z}^{*,q}_{Spitzweck}$ on $\mathbf{Sch}_k^{qcqs}$.

3. A suitable extension of the Beilinson-Lichtenbaum axioms hold for $X \in \mathbf{Sch}_k^{qcqs}$:

The motivic spectral sequence converging to Bass-Quillen K theory (in progress in $\mathrm{char}\ 0$), the relation of the mod m theory with the étale theory, and in characteristic $p > 0$,

$$\mathbb{Z}/p(q)^{\mathrm{mot}}(X) = R\Gamma_{\mathrm{cdh}}(X, \Omega^q_{\log})[-q] \times_{R\Gamma_{\mathrm{eh}}(X,\Omega^q_{\log})[-q]} \mathbb{Z}/p(q)^{\mathrm{syn}}(X)$$

Non-$\mathbb{A}^1$-invariant motivic cohomology

Kelly-Saito have a candidate for motivic complexes $\mathbb{Z}(q)^{mot}$ on $\mathbf{Sch}^{qcqs}$

$$\mathbb{Z}(q)^{mot} := L_{pro-cdh}\tilde{\mathbb{Z}}(q)$$

There is work on-going to show the two definitions agree.

Non-$\mathbb{A}^1$-invariant motivic categories

At present, the theory $\mathbb{Z}(*)^{\mathrm{mot}}$ does not have a categorical framework. Some possibilities:

• The theory of "motives with modulus" by Kahn, Miyazaki, Saito, Yamazaki. They replace Cor_k with a correspondence category built out of pairs (X, D), D a Cartier divisor on X, and replacing $\mathbb{A}^1$-invariance with $(\mathbb{P}^1, \infty)$-invariance.

• Binda-Park-Østvær have adapted Voevodsky's $\mathrm{DM}(k)$ to log-schemes, giving a category of "log motives" $\mathrm{DM}_{\log}(k)$, that replaces $\mathbb{A}^1$-invariance with $(\mathbb{P}^1, \infty)_{\log}$ invariance.

• Annala-Iwasa have constructed a non-$\mathbb{A}^1$-invariant motivic stable homotopy theory adapted to Bass-Quillen K-theory. Perhaps a modification could be a home for $\mathbb{Z}(*)^{\mathrm{mot}}$?

The quadratic theory

Milnor-Witt K-theory

Milnor-Witt K-theory and Morel's theorem on the motivic 0-stem form the basis of the connection of motivic homotopy theory with quadratic forms.

Hopkins-Morel have defined the *Milnor-Witt* K-theory of a field F, $K_*^{MW}(F)$, via generators:

- $[u] \in K_1^{MW}(F)$, for $u \in F \setminus \{0\}$,
- $\eta \in K_{-1}^{MW}(F)$,

and relations:

- $[u]\eta = \eta[u]$
- $[u][1-u] = 0$ for $u \in F \setminus \{0,1\}$
- $[uv] = [u] + [v] + \eta[u][v]$
- $\eta(2 + \eta[-1]) = 0$

Morel's theorem

Morel shows:

- The $K_n^{MW}(F)$ for fields extend to a sheaf of graded rings $\mathcal{K}_*^{MW}$ on $\mathbf{Sm}_k$.
- $\mathcal{K}_0^{MW}$ is the sheaf $\mathcal{GW}$ of Grothendieck-Witt rings, and for $n < 0$, $\mathcal{K}_n^{MW}$ is the sheaf of Witt groups.
- For $n \geq 0$, there is an exact sequence

$$0 \to \mathcal{I}^{n+1} \to \mathcal{K}_n^{MW} \to \mathcal{K}_n^{M} \to 0; \ \mathcal{I} = \ker(\mathrm{rank} : \mathcal{GW} \to \mathbb{Z}).$$

Morel's theorem connects $\mathcal{K}_*^{MW}$ with motivic homotopy theory.

Theorem (Morel)

The homotopy sheaves $\pi_{n,n}\mathbb{S}_k$ of the motivic sphere spectrum over k are given by Milnor-Witt K-theory:

$$\pi_{n,n}\mathbb{S}_k = \mathcal{K}_{-n}^{MW}, \ \ n \in \mathbb{Z}.$$

In particular

$$\boxed{\mathrm{End}_{\mathsf{SH}(k)}(\mathbb{S}_k) = \pi_{0,0}\mathbb{S}_k(k) = \mathsf{GW}(k)}$$

Chow-Witt groups

Bloch's formula (proven by Quillen for $X \in \mathbf{Sm}_k$)

$$\mathrm{CH}^n(X) = H^n(X, \mathcal{K}_n^M)$$

suggests the following definition of the Chow-Witt groups, introduced by Barge-Morel and developed by Fasel.

Definition (Chow-Witt groups)

The *n*th Chow-Witt group on $X \in \mathbf{Sm}_k$ is

$$\widetilde{\mathrm{CH}}^n(X) := H^n(X, \mathcal{K}_n^{MW})$$

Chow-Witt groups

- Elements of $\widetilde{\mathrm{CH}}^n(X)$ can be described as codimension n "quadratic cycles"

$$x = \sum_i q_i Z_i,\ \mathrm{codim}_X Z_i = n,\ q_i \in \mathrm{GW}(k(Z_i)),\ \partial(x) = 0,$$

  modulo a Milnor-Witt refinement of rational equivalence.
- There is a quadratic intersection theory, quadratic Euler classes for vector bundles, quadratic virtual fundamental classes, ... (Fasel, Déglise-Jin-Khan, Kass-Wickelgren, L., ...)
- The "quadratic degree" for an enumerative problem over k lives in $\mathrm{GW}(k)$. Its rank yields the classical count and for $\sigma : k \hookrightarrow \mathbb{R}$, the signature yields a signed count of the real solutions.

Milnor-Witt motivic cohomology

The Chow-Witt theory is represented in $\mathrm{SH}(k)$ by a spectrum $\widetilde{H\mathbb{Z}}$.

Definition (Bachmann)

Let $\widetilde{H\mathbb{Z}} = f_0 \tau_0 \mathbb{S}_k$.

Here τ_0 is the trunction to the heart for Morel's homotopy t-structure, via the "usual" Postnikov tower in $\mathrm{SH}(k)$ measuring S^1-connectivity.

Theorem (Bachmann-Fasel)

$$\widetilde{H\mathbb{Z}}^{2n,n}(X) \cong \widetilde{\mathrm{CH}}^n(X)$$

Compare with $H\mathbb{Z}^{2n,n}(X) = H^{2n}(X, \mathbb{Z}(n)) \cong \mathrm{CH}^n(X)$.

Taking the full bi-graded theory gives us *Milnor-Witt motivic cohomology* $X \mapsto \widetilde{H\mathbb{Z}}^{p,q}(X)$.

Milnor-Witt motives

Bachmann-Calmès-Déglise-Fasel-Østvær put this in a categorical framework with their category of Milnor-Witt motives $\widetilde{\mathrm{DM}}(k)$.

They refine Voevodsky's category Cor_k of finite correspondences to "quadratic finite correspondences" $\widetilde{\mathrm{Cor}}_k$:

$$\widetilde{\mathrm{Cor}}_k(X, Y) = \{z := \sum_i q_i Z_i \mid Z_i \subset X \times Y,\ Z_i \to X \text{ finite, surjective}, \\ q_i \in \mathrm{GW}(k(Z_i), \omega_{Y/k}),\ \partial(z) = 0\}$$

Following Voevodsky's playbook, this yields a triangulated tensor category $\widetilde{\mathrm{DM}}(k)$, with Tate objects $\tilde{\mathbb{Z}}(q)$, motives $\tilde{M}(X)$ for $X \in \mathbf{Sm}_k$.

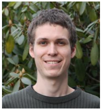
The Milnor-Witt motivic cohomology is represented in $\widetilde{\mathrm{DM}}(k)$:

$$\widetilde{H\mathbb{Z}}^{p,q}(X) \cong \mathrm{Hom}_{\widetilde{\mathrm{DM}}(k)}(\tilde{M}(X), \tilde{\mathbb{Z}}(q)[p])$$

Motivic cohomology: The next generation


Binda


Yakerson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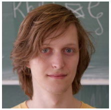
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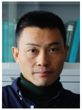
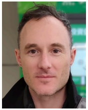

Yamazaki


Morrow


Bhatt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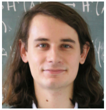
Scholze


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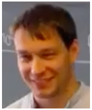
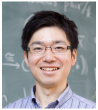
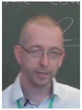

Bachmann


Elmanto


Wickelgren


Miyazaki


Spitzweck


Wendt


Röndigs


Østvær


Druzhinin


Tamme


Land

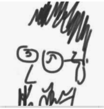

Ananyevskiy


Kerz


Déglise


Cisinski


Pauli


Calmès


Fasel


Annala


Iwasa


Khan


Haesemeyer


Strunk

Thank you!