Spin systems with hyperbolic symmetry

Roland Bauerschmidt

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Based on joint works with N. Crawford, T. Helmuth, A. Swan.

UNIVERSITY OF CAMBRIDGE

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A. Motivation and background:

- Anderson transition
- Classic Heisenberg model

B. Focus on two related statistical physics models:

- Linearly reinforced walks
- Random forests

C. Relation to hyperbolic sigma models.

D. What we understand and some of the many things we do not.

Given $G = (\Lambda, E) \subset \mathbb{Z}^d$ let Δ be the standard Laplacian matrix: $(\Delta f)_x = \sum_{xy \in E} (f_y - f_x)$.

Random Schrödinger operator, or random band matrices:

$$H_\lambda = -\Delta + \lambda V, \quad (V = (V_x)_{x\in\Lambda} \text{ i.i.d. normal}), \qquad \text{or} \qquad H_\lambda = \Big[\ 1/\lambda\ \Big]_{x,y\in\Lambda}$$

Interested in $d \geqslant 2$. The parameter λ should be independent of the size of Λ.

Metal–insulator transition: *Are the eigenfunctions extended (metallic phase) or localised (insulating phase) at a given energy E (point in spectrum), depending on λ and d?*

Information encoded in two-point function (with $\eta \downarrow 0$):

$$G_{\lambda,\eta}(x,y) = \mathbb{E}|(H_\lambda - E - i\eta)^{-1}(x,y)|^2$$

The absolute value on the RHS is crucial.

[Wegner, Efetov, Zirnbauer, ...]: Problem can be related exactly to (complicated) spin system with hyperbolic symmetry and supersymmetry. Much physics understanding based on this.

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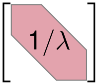

$x,y \in \Lambda$

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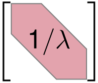

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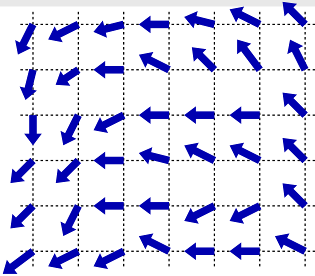
Background 2: classical Heisenberg model

Spin configurations are $u : \Lambda \to \mathbb{S}^2$, i.e., $u = (u_x)_{x\in\Lambda}$ with $u_x \in \mathbb{S}^2$.



Classical Heisenberg model (ferromagnet):

$$\langle F(u)\rangle_\beta \propto \int_{(\mathbb{S}^2)^\Lambda} \prod_{x\in\Lambda} du_x \ \exp\Bigg(\beta \sum_{xy\in E} \underbrace{(u_x \cdot u_y - 1)}_{-\frac{1}{2}|u_x - u_y|^2}\Bigg) F(u).$$

Magnetism: *Do spins align over long distances (ferromagnetic phase), depending on* β *and* d*?*

Information encoded in two-point correlation functions:

$$G_\beta(x,y) = \frac{1}{3}\langle u_x \cdot u_y\rangle_\beta$$

- Neighbouring spins u_x and u_y prefer to point in the same direction (if β is large).
- (Lattice) non-linear sigma model with target $\mathbb{S}^2$, i.e., $O(3)$ symmetry — nonabelian.

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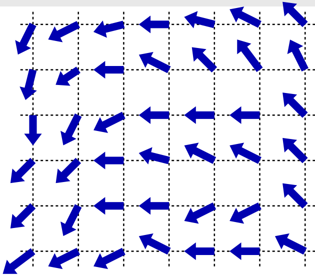

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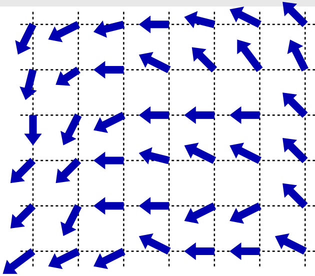

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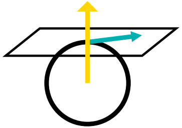
(symmetric and transversal two-point functions)



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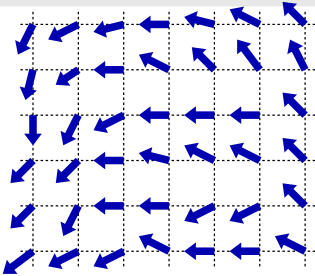

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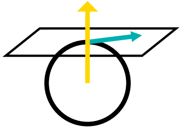

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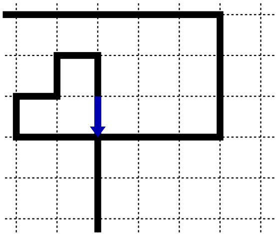
Model 1: linearly reinforced random walk

Continuous parameter walk $(X_t)_{t \geqslant 0} \subset \Lambda$.



Vertex-reinforced jump process (VRJP):

$$\mathbb{P}_\beta\Big(X_{t+dt} = y \Big| X_t = x, (X_s)_{s<t}\Big) = \beta\, \mathbf{1}(xy \in E) \left(1 + \int_0^t \mathbf{1}\{X_s = y\}\, ds\right) dt$$

[VRJP: Werner, Davis–Volkov, Sabot–Tarres, ... ERRW: Diaconis, Kean, Rolles, Merkl, ...]

- History-dependent random walk that prefers to revisit sites visited before.
- Parameter $1/\beta$ is the reinforcement strength.

How does the behaviour of the walk depend on β and d? Diffusion–localisation transition?

$$G_{\beta,h}(x,y) = \int_0^\infty \mathbb{P}_\beta(X_t = y | X_0 = x)\, e^{-ht}\, dt$$

Exactly equal to two-point function of non-linear sigma model with target $\mathbb{H}^{2|2}$ (later).

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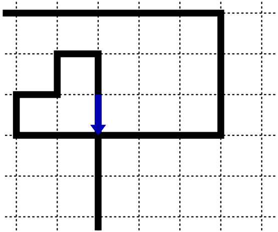

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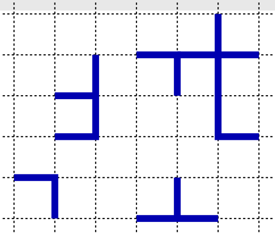
Model 2: random forests

Given $G = (\Lambda, E)$, a forest $F = (\Lambda, E(F))$ is a subgraph without cycles.



Arboreal gas:

$$\mathbb{P}_\beta(F) \propto \beta^{|E(F)|} \mathbf{1}(F \text{ is a forest}).$$

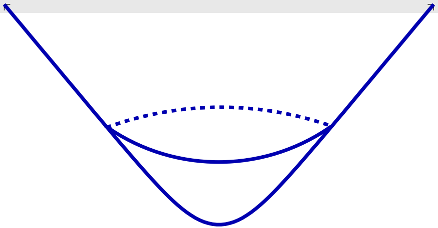
[Fortuin–Kasteleyn, Lubensky–Isaacson, Pemantle, Kahn, Grimmett, Caracciolo et al., ...]

- Activity β controls density of the forest: $\beta \to \infty$ gives uniform spanning tree (fully packed).
- Model for gelation of polymers [Lubensky–Isaacson]. Other interpretations:
  - Uniform measure on spanning forests (on β weighted graph).
  - Bernoulli bond percolation with edge probability $p = \beta/(1+\beta)$ conditioned to have no cycles.
  - Limit $q \to 0$ with $p = \beta q$ of random cluster representation of q-state Potts model.
  - Hardcore gas of lattice trees with activity β (thus the name arboreal gas).

Is there a macroscopic (extended) tree or are all trees small (localised) given β and d?

$$G_\beta(x, y) = \mathbb{P}_\beta(x \leftrightarrow y)$$

Exactly equal to two-point function of non-linear sigma model with target $\mathbb{H}^{0|2}$ (later).



Vectors: $u_x = (z_x, v_x, w_x) \in \mathbb{R}^3 = \mathbb{R}^{1+2}$ for each $x \in \Lambda$

Minkowski inner product: $u_x \cdot u_y = v_x v_y + w_x w_y - z_x z_y$

$$u_x \cdot u_x = -1, \quad z_x > 0 \quad \rightsquigarrow \text{Hyperbolic plane } \mathbb{H}^2 \text{ (symmetric space)}$$

Concretely: $z_x = \sqrt{1 + v_x^2 + w_x^2}$ and the $SO^+(2,1)$ invariant integral on $\mathbb{H}^2$ is

$$\int_{\mathbb{H}^2} du_x\, F(u_x) = \int_{\mathbb{R}^2} dv_x\, dw_x \frac{1}{\sqrt{1 + v_x^2 + w_x^2}} F(v_x, w_x, \sqrt{1 + v_x^2 + w_x^2}).$$

Hyperbolic sigma model:

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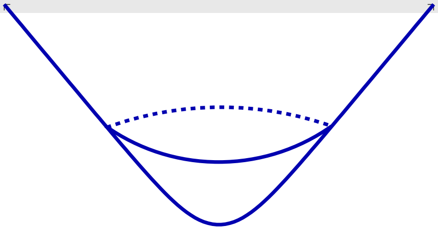
[Spencer–Zirnbauer]



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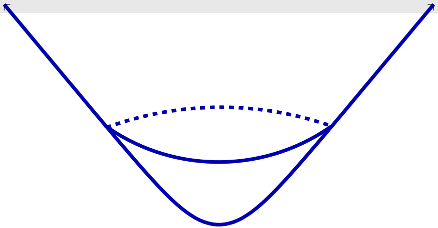
[Spencer–Zirnbauer]

Hyperbolic plane $\mathbb{H}^2$



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[Spencer–Zirnbauer]

Fermionic hyperbolic plane $\mathbb{H}^{0|2}$

Supervectors $u_x = (z_x, \xi_x, \eta_x) \in \mathbb{R}^{1|2}$ for each $x \in \Lambda$ — the ξ_x and η_x anticommute, z_x commutes

Formal inner product: $u_x \cdot u_y = \eta_x \xi_y + \eta_y \xi_x - z_x z_y$

$$u_x \cdot u_x = -1 \qquad z_x > 0$$

$\rightsquigarrow$ Fermionic hyperbolic plane $\mathbb{H}^{0|2}$ (symmetric superspace)

Concretely: $z_x = \sqrt{1 + 2\eta_x \xi_x} = 1 - \xi_x \eta_x$ and the $OSp(1|2)$ invariant superintegral is

$$\int_{\mathbb{H}^{0|2}} du_x\, F(u_x) = \int \partial_{\eta_x} \partial_{\xi_x} \frac{1}{\sqrt{1 + 2\eta_x \xi_x}} F(\xi, \eta, \sqrt{1 + 2\eta_x \xi_x})$$

Fermionic hyperbolic sigma model:

$$\langle F(u) \rangle_{\beta, h} \propto \int_{(\mathbb{H}^{0|2})^\Lambda} \prod_{x \in \Lambda} du_x\, \exp\left(\beta \sum_{xy \in E} \underbrace{(u_x \cdot u_y + 1)}_{-\frac{1}{2}|u_x - u_y|^2} - h \sum_{x \in \Lambda} z_x \right) F(u).$$

[B–Crawford–Helmuth–Swan]

Hyperbolic superplane $\mathbb{H}^{2|2}$

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Formal inner product: $u_x \cdot u_y = v_x v_y + w_x w_y + \eta_x \xi_y + \eta_y \xi_x - z_x z_y$

$$\begin{aligned} u_x \cdot u_x &= -1 \\ z_x &> 0 \end{aligned} \quad \rightsquigarrow \text{Hyperbolic superplane } \mathbb{H}^{2|2} \text{ (symmetric superspace)}$$

Concretely: $z_x = \sqrt{1 + v_x^2 + w_x^2 + 2\eta_x\xi_x}$ and the $OSp(2,1|2)$ invariant superintegral is

$$\int_{\mathbb{H}^{2|2}} du_x \, F(u_x) = \int dv_x \, dw_x \, \partial_{\eta_x} \partial_{\xi_x} \frac{1}{\sqrt{1 + v_x^2 + w_x^2 + 2\eta_x\xi_x}} F(v_x, w_x, \xi_x, \eta_x, \sqrt{1 + v_x^2 + w_x^2 + 2\eta_x\xi_x})$$

Supersymmetric hyperbolic sigma model:

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[Zirnbauer, Disertori–Spencer–Zirnbauer]

Hyperbolic sigma models $\longleftrightarrow$ statistical physics

$\mathbb{H}^{2|2}$ $\longleftrightarrow$ VRJP [Sabot–Tarres, B–Helmuth–Swan]. Example:

$$\langle v_x v_y \rangle_{\beta,h}^{\mathbb{H}^{2|2}} = \int_0^\infty \mathbb{P}_\beta(X_t = y | X_0 = x)\, e^{-ht}\, dt$$

$\mathbb{H}^{0|2}$ $\longleftrightarrow$ Arboreal Gas [Caracciolo–Jacobsen–Saleur–Sokal–Sportiello, B–Crawford–Helmuth–Swan]. Example:

$$-\langle u_x \cdot u_y \rangle_{\beta}^{\mathbb{H}^{0|2}} = \mathbb{P}_\beta(x \leftrightarrow y)$$

$$\langle \eta_x \xi_y \rangle_{\beta,h}^{\mathbb{H}^{0|2}} = \mathbb{P}_{\beta,h}(x \leftrightarrow y, x \not\leftrightarrow \partial)$$

Symmetries $\longleftrightarrow$ constraints. Example:

$$\sum_y \langle v_x v_y \rangle_{\beta,h}^{\mathbb{H}^{2|2}} = \frac{1}{h} \langle z_x \rangle_{\beta,h}^{\mathbb{H}^{2|2}} = \frac{1}{h} \leftrightarrow \text{conservation of probability}$$

Analogously in random matrix models: Ward identity $\leftrightarrow$ quantum sum rule.

Conjectured phase diagram: similar for all of these models

Example: Heisenberg model

- High temperature $\beta \ll 1$ (or $\lambda \gg 1$), dimensions $d \geqslant 1$:

  No long-range order and exponential decay of correlations:

$$\langle u_x \cdot u_y \rangle_\beta = O_\beta(e^{-c_\beta |x-y|})$$

- Low temperature $\beta \gg 1$ (or $\lambda \ll 1$), dimensions $d \geqslant 3$:

  Long-range order and diffusive correlations [Goldstone]:

$$\langle u_x \cdot u_y \rangle_\beta \approx m_\beta^2 - \frac{c_\beta}{|x-y|^{d-2}}$$

- All temperatures $\beta > 0$, dimension $d = 2$:

  No long-range order [Mermin–Wagner] and exponential decay of correlations [Polyakov, Friedan]:

$$\langle u_x \cdot u_y \rangle_\beta = O_\beta(e^{-c_\beta |x-y|}), \qquad c_\beta \approx e^{-c\beta}$$

The role of symmetry is crucial at low temperature.

Phase diagram: What do we know?

- High temperature / large disorder is well understood in all cases in any $d \geqslant 1$:
  - Heisenberg model: spins are approximately independent [high temperature expansion]
  - VRJP is localised [Disertori–Spencer, Sabot–Tarres, Angel–Crawford–Kozma, Collevecchio–Zeng]
  - Arboreal gas consists of small trees [domination by Bernoulli percolation]
  - Random matrix models: eigenfunctions are localised [Fröhlich–Spencer, Aizenman–Molchanov]
- Low temperature / weak disorder: expect long-range order with diffusive correlations in $d \geqslant 3$.
  - Heisenberg model: long-range order with spin wave correlations [Fröhlich–Simon–Spencer, Balaban]
  - VRJP is quasi diffusive [Disertori–Spencer–Zirnbauer, Sabot–Tarres]
  - Arboreal gas percolates with diffusive correlations [B–Crawford–Helmuth]
  - Random matrix models: open
- Two dimensions: expect exponential decay at all temperatures / any disorder:
  - Heisenberg model: no magnetisation, polynomial bounds [Mermin–Wagner, McBryan–Spencer]
  - VRJP: recurrent, polynomial bounds [Merkl–Rolles, B–Helmuth–Swan, Sabot, Sabot–Zeng Peled–Kozma]
  - Arboreal: no percolation, polynomial bounds on correlations [B–Crawford–Helmuth–Swan]
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  - VRJP is quasi diffusive [Disertori–Spencer–Zirnbauer, Sabot–Tarres]
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  - Random matrix models: open
- Two dimensions: expect exponential decay at all temperatures / any disorder:
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  - VRJP: recurrent, polynomial bounds [Merkl–Rolles, B–Helmuth–Swan, Sabot, Sabot–Zeng Peled–Kozma]
  - Arboreal: no percolation, polynomial bounds on correlations [B–Crawford–Helmuth–Swan]
  - Random matrix models: open

Phase diagram: What do we know?

- High temperature / large disorder is well understood in all cases in any $d \geqslant 1$:
  - Heisenberg model: spins are approximately independent [high temperature expansion]
  - VRJP is localised [Disertori–Spencer, Sabot–Tarres, Angel–Crawford–Kozma, Collevecchio–Zeng]
  - Arboreal gas consists of small trees [domination by Bernoulli percolation]
  - Random matrix models: eigenfunctions are localised [Fröhlich–Spencer, Aizenman–Molchanov]
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Example: Arboreal gas

Fact (easy). Let $d \geqslant 1$. For $\beta < p_c/(1-p_c)$ with p_c the critical Bernoulli percolation parameter,

$$\mathbb{P}_\beta(x \leftrightarrow y) \leqslant C_\beta e^{-c_\beta |x-y|}.$$

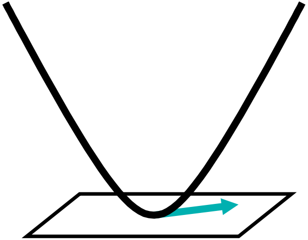
Theorem (B–Crawford–Helmuth–Swan, Sabot). Let $d = 2$. Then for all $\beta > 0$,

$$\mathbb{P}_\beta(x \leftrightarrow y) \leqslant C_\beta |x-y|^{-c_\beta}.$$

Theorem (B–Crawford–Helmuth). Let $d \geqslant 3$. Then, for $\beta \geqslant \beta_0$,

$$\mathbb{P}_\beta(x \leftrightarrow y) = \theta(\beta)^2 - \frac{c_\beta}{\beta|x-y|^{d-2}} + O\left(\frac{1}{\beta|x-y|^{d-2+\kappa}}\right) + O\left(\frac{1}{\beta\,\mathrm{diam}(\Lambda)^\kappa}\right)$$

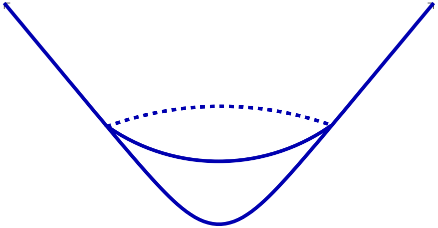
and $\theta(\beta) = 1 - O(1/\beta)$, for Λ a discrete torus of side L^N, large L fixed.



Horospherical coordinates for $\mathbb{H}^{n|2m}$

For $\mathbb{H}^2$ horospherical coordinates are $(t, s) \in \mathbb{R}^2$ such that:

$$z = \cosh t + \frac{1}{2} e^t s^2$$
$$v = \sinh t - \frac{1}{2} e^t s^2$$
$$w = e^t s$$



- The energy becomes:

$$\frac{1}{2} \sum_{xy} (-u_x \cdot u_y - 1) = \sum_{xy} \cosh(t_x - t_y) + \frac{1}{2} \sum_{xy} e^{t_x + t_y} (s_x - s_y)^2 .$$

- The s variable appears quadratically — Gaussian when conditioned on t-variable.
- Generalises to higher dimensions and the hyperbolic superspaces $\mathbb{H}^{n|2m}$.

Magic formula for ERRW/VRJP

- [Diaconis et al.]: Edge-reinforced random walk (ERRW) is equivalent to a random walk in random environment. The environment is given by magic formula.
- [Sabot–Tarres]: Same for VRJP (actually generalising ERRW):

$$\mathbb{P}_{\mathrm{VRJP}(\beta)} \propto \int \mathbb{P}_{\mathrm{SRW}(\beta(t))} \, \nu_\beta(dt)$$

  The law of the random environment (conductances) is given by version of magic formula:

$$\beta_{xy}(t) = \beta e^{t_x + t_y}$$

$$\nu_\beta(dt) \propto e^{-\beta \sum_{xy \in E} \cosh(t_x - t_y)} (\det{}^0(-\Delta_{\beta(t)}))^{1/2} \prod_{x \neq 0} e^{-t_x} \, dt_x$$

  This is exactly the t-marginal of the horospherical coordinates of the $\mathbb{H}^{2|2}$ model! Why?
- [B–Helmuth–Swan]: Relation explained by combining:
  - Duality (BFS–Dynkin isomorphism) between generators of hyperbolic rotations and of the VRJP
  - Supersymmetric localisation ($\sim$ Duistermaat–Heckman formula)

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Magic formula for arboreal gas

- [B–Crawford–Helmuth–Swan]: Similar magic formula holds for arboreal gas

$$\mathbb{P}_\beta(0 \leftrightarrow x) \propto \int_{\mathbb{R}^{\Lambda \setminus 0}} e^{t_x} e^{-\beta \sum_{xy \in E} \cosh(t_x - t_y)} (\det{}^0(-\Delta_{\beta(t)}))^{3/2} \prod_{x \neq 0} e^{-3t_x} \, dt_x.$$

- Integrand is the **same** as magic formula for VRJP except that power $1/2$ is replaced by $3/2$.

  Corresponds to change of dimension $\mathbb{H}^{2|2} \to \mathbb{H}^{2|4}$ in horospherical coordinates.

What are the magic formulas good for?

- Different (probabilistic) perspective on original model (VRJP, arboreal gas).
- Similar in spirit to relation Ising/Potts model to random cluster model, and other examples.
- Important for various results obtained for these models, but not a panacea.

Many questions: Correlation inequalities?

- Arboreal gas.
  - Conjecture [Pemantle, Kahn, Grimmett]: Negative edge correlation holds

    $$\mathbb{P}_\beta[e_1 \in F, e_2 \in F] \leqslant \mathbb{P}_\beta[e_1 \in F]\mathbb{P}_\beta[e_2 \in F] \qquad \text{for all edges } e_1, e_2.$$

    Equivalently, the two-point function is monotone in (edge-dependent) β.
  - Theorem [Bränden–Huh]:

    $$\mathbb{P}_\beta[e_1 \in F, e_2 \in F] \leqslant 2\,\mathbb{P}_\beta[e_1 \in F]\mathbb{P}_\beta[e_2 \in F].$$
- Heisenberg model.
  - Conjecture [second Griffith inequality]:

    $$\langle (u_{x_1} \cdot u_{y_1})(u_{x_2} \cdot u_{y_2}) \rangle_\beta \geqslant \langle u_{x_1} \cdot u_{y_1} \rangle_\beta \langle u_{x_2} \cdot u_{y_2} \rangle_\beta$$

    Equivalently, the two-point function is monotone in (edge-dependent) β.
- Vertex-reinforced jump process.
  - Theorem [Poudevigne]: the two-point function is monotone in (edge-dependent) β.

Many questions: Low temperature universality?

Expect that continuous symmetries of spin models lead to rich low temperature phases of statistical physics models:

- Arboreal gas: is the cluster size distribution universal? Distributions known on complete graph [Luczak–Pittel, Martin–Yeo].
- Heisenberg model: in the cycle representation (not discussed), are the cycle size distributions universal and given by Poisson–Dirichlet statistics? [Schramm, Ueltschi, Nahum–Chalker, ...]
- Random matrix models: are the eigenvalue distributions universal and given by Wigner–Dyson statistics? In mean-field settings understood [Erdős, Yau, ...]; also [Shcherbina–Shcherbina, ...].

Many questions: Critical behaviour?

- Expect differences between
  - compact target space: Heisenberg model, Arboreal gas (effectively)
  - noncompact target space: VRJP, random matrix models
- What are the upper critical dimensions?
  - Heisenberg model: undisputed that $d_c = 4$
  - Arboreal gas: expect $d_c = 6$ (evidence from mean-field theory, numerics, and field theory [Klebanov])
  - VRJP and random matrix models: general belief that $d_c = \infty$?
- Multiple upper critical dimensions (for different observables)?
- Critical exponents? Multifractal behaviour?

Many questions: Models with related motivation?

- Network models ($\sim$ Quantum Hall transition) [Chalker–Coddington, Beamond–Owczarek–Cardy, ...]
- Random permutation models ($\sim$ Quantum Heisenberg model) [Toth, Schramm, ...]
- ...

Rich models with interesting behaviour beyond the standard examples in statistical physics, but very little is understood.

Also see reviews by J. Cardy and T. Spencer.

Thank you!