# Probability theory for random groups arising in number theory

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# Overview

1. Motivating examples of random groups
2. The moment problem for random groups
3. Universality for random groups

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   Related: what is the distribution of the Jacobian of an Erdős–Rényi random graph (i.e. sandpile group, i.e. $\mathbb{Z}^n / \Delta(\mathbb{Z}^n)$ where $\Delta$ is the graph Laplacian)

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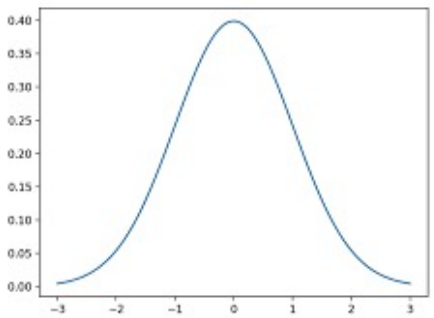

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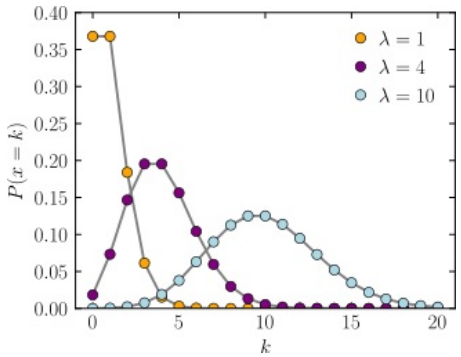


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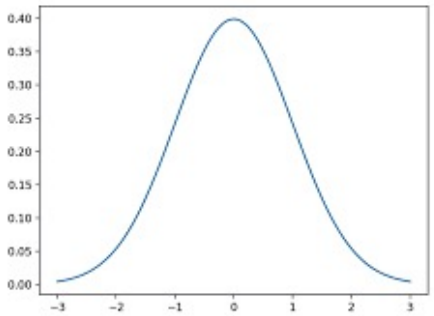

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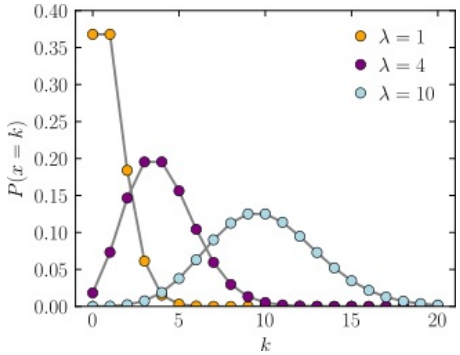


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Knowledge of $\mathbb{E}[X]$ and these up to $k$ is equivalent to knowledge of first $k$ moments

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Moments are often much more accessible than direct information about the distribution

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*Let $X, Y$ be random finite abelian groups. If for each finite ab. group $A$,*

$$\mathbb{E}(\#\operatorname{Sur}(X, A)) = \mathbb{E}(\#\operatorname{Sur}(Y, A)) = O(|\wedge^2 A|),$$

*then $X$ and $Y$ have the same distribution.*

Application: (Ellenberg, Venkatesh, Westerland '16; Liu, W., Zureick-Brown '19) to $\operatorname{Pic}(C)$ of curves $C$ over finite fields $\mathbb{F}_q$, function field Cohen-Lenstra-Martinet for $q \to \infty$
(EVW applied their own version for $p$-Sylow subgroups, LWZB applied analog for finite abelian $R$-modules)

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$$\text{Prob}(\pi_1(M) \text{ has no non-trivial } S\text{-group quotients})$$
$$= \prod_{p \in S} \prod_{j \geq 1} (1 + p^{-j})^{-1} \prod_{\substack{N \text{ non-ab. finite} \\ \text{simple } S\text{-group}}} e^{-\frac{|H_2(N,\mathbb{Z})|}{|\text{Out}(N)|}}.$$

# Overview

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The output distribution of this process that combines the $X_i$ is largely *insensitive* to the input distribution.

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Applications (of analogs for symmetric matrices):
Jacobians/sandpile groups of random graphs W. '17 (Erdős–Rényi), Mészáros '20 (uniform $d$-regular graphs $\implies$ non-singularity of adjacency matrix /$\mathbb{R}$)

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