Real Gromov-Witten theory

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International Congress of Mathematicians
12 July 2022

Symplectic manifolds and pseudo-holomorphic maps

- A symplectic manifold (X, ω) is a smooth manifold X together with a closed non-degenerate 2-form ω.
  Examples: Cotangent bundles T^*M, Kähler manifolds (e.g. $\mathbb{CP}^n$).
- On a symplectic manifold (X, ω) there exists an infinite dimensional family of almost-complex structures

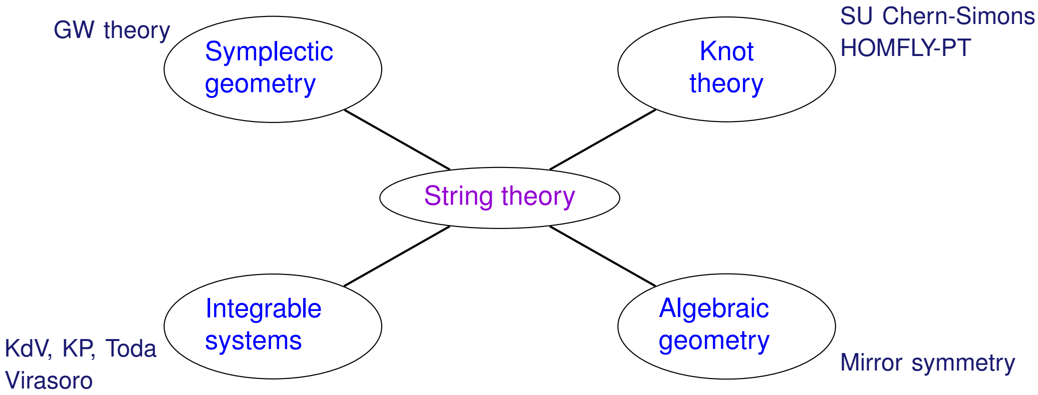
$$J : TX \longrightarrow TX, \quad J^2 = -Id.$$

- A pseudo-holomorphic map is a map from a Riemann surface (Σ_g, j) to X

$$u : (\Sigma_g, j) \longrightarrow (X, J), \qquad J \circ du = du \circ j.$$

Connections motivated by physics

Witten : The GW potential is a partition function of a (type II A) topological string theory.



Moduli spaces and Gromov-Witten invariants

Let (X,ω) be a symplectic manifold and J a compatible almost complex structure.

- Moduli space of stable marked pseudo-holomorphic maps

$$\overline{\mathcal{M}}_{g,l}(X,A) = \left\{ u : (\Sigma_g, j, x_1, .., x_l) \to X, [u] = A \in H_2(X), J \circ du = du \circ j \right\} / Diff.$$

  Compactification : including maps from nodal domains.

- There are natural maps

$$ev_i : \overline{\mathcal{M}}_{g,l}(X,A) \longrightarrow X, \quad [u,j,x_1,..,x_l] \mapsto u(x_i).$$

- The Gromov-Witten invariants are defined as

$$GW_g^{X,A}(\alpha_1,\dots,\alpha_l) = \int_{\overline{\mathcal{M}}_{g,l}(X,A)} \wedge_i ev_i^* \alpha_i,$$

  where α_i representatives of cohomology classes on X.

GW generating series

- The Gromov-Witten potential is a formal power series in formal variables keeping track of the genus, the homology class of the map, and the cohomology classes of the constraints.
- Examples :
  - Generating function of a Calabi-Yau 3-fold :
    $$GW(u,q) = \sum GW_g^{X,A} q^A u^{2-2g} \quad \text{and} \quad Z = \exp(GW(q,u)).$$
  - Generating function $GW_{g=0}(\mathbb{CP}^2) = \sum_d N_d q^d$ - generating series of the counts N_d of genus 0 degree d curves in $\mathbb{CP}^2$ passing through $3d-1$ points.

$$N_1 = 1 \quad N_2 = 1 \quad N_3 = 12 \quad N_4 = 620 \quad N_5 = 87304 \quad \ldots$$

Real enumerative geometry

A typical question in enumerative geometry :

- Let $p_1, \ldots, p_8$ be eight points in $\mathbb{KP}^2$. How many rational cubics in $\mathbb{KP}^2$ pass through these points?

  Answer : 12 if $\mathbb{K} = \mathbb{C}$, $\quad$ $8, 10, 12$ if $\mathbb{K} = \mathbb{R}$.

In the real case an invariant number can only give a lower bound on the number of objects and they should be counted with a sign.

- First invariant count was obtained by **Welschinger** in 2003 (g=0, dim=4,6).

$$N_1 = 1 \quad N_2 = 1 \quad N_3 = 12 \quad N_4 = 620 \quad N_5 = 87304 \quad \ldots$$
$$N_1^{\mathbb{R}} = 1 \quad N_2^{\mathbb{R}} = 1 \quad N_3^{\mathbb{R}} = 8 \quad N_4^{\mathbb{R}} = 240 \quad N_5^{\mathbb{R}} = 18264 \quad \ldots$$

Real moduli spaces

- (X, ω, ϕ) a symplectic manifold with an involution $\phi : X \to X$ satisfying $\phi^*\omega = -\omega$.
  Example: $\mathbb{CP}^n$ with the standard conjugation.

- The maps invariant under ϕ form the real moduli space $\overline{\mathcal{M}}_g^{\phi}(X, A)$.

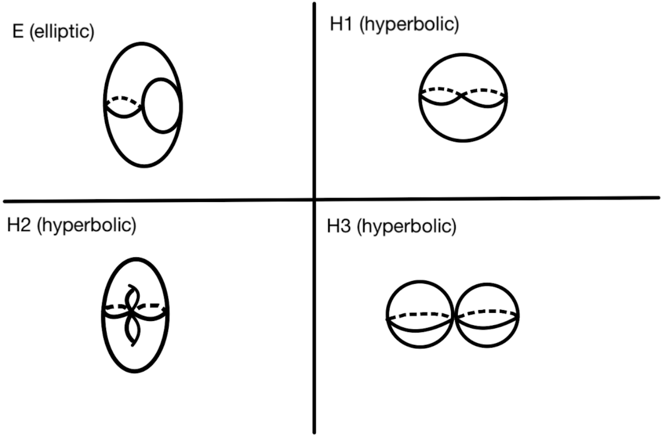
$$\overline{\mathcal{M}}_g^{\phi}(X, A) = \bigcup_{\sigma} \overline{\mathcal{M}}_g^{\phi,\sigma}(X, A),$$

  where σ is an orientation-reversing involution on the domain and

$$\overline{\mathcal{M}}_g^{\phi,\sigma}(X, A) = \{u : \Sigma_g \to X, [u] = A \in H_2(X), J \circ du = du \circ j, \phi \circ u = u \circ \sigma\} / Diff.$$

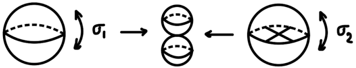
- Compactification : including maps from **symmetric** nodal domains.

The four types of codimension 1 strata

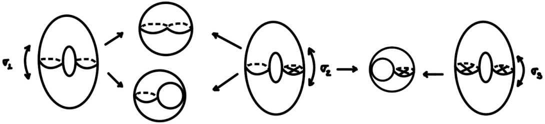


Boundary vs hyperplane

- The moduli spaces $\overline{\mathcal{M}}_g^{\phi,\sigma}(X, A)$ have boundary.
  - ∘ genus 0 degeneration and boundary for the two types of involutions



  - ∘ genus 1 degeneration and boundary for the three types of involutions



- The moduli space $\overline{\mathcal{M}}_g^{\phi}(X, A)$ does not have boundary. Thus, when this space is orientable, it gives rise to real Gromov-Witten invariants.

Real Gromov-Witten invariants in arbitrary genus

Theorem (G.-Zinger)

Let (X, ω, ϕ) be a symplectic manifold with anti-symplectic involution ϕ and $\dim X = 2(2n+1)$. If there exists a Real line bundle $(L, \tilde{\phi}) \longrightarrow (X, \phi)$ such that

$$\Lambda^{top}(TX, d\phi) \cong (L, \tilde{\phi})^{\otimes 2} \quad \textit{and} \quad TX^{\phi} \oplus 2L^{\tilde{\phi}} \textit{ is spin,}$$

then the real moduli spaces $\overline{\mathcal{M}}_g^{\phi}(X, A)$ are orientable. As a consequence, the numbers

$$RGW_g^{X,\phi,A}(\alpha_1, \ldots, \alpha_l)$$

are well-defined invariants of the triple (X, ω, ϕ).

Examples: $\mathbb{CP}^{2n+1}$, many complete intersections and Calabi-Yau manifolds.

Computation in genus 0

There are different methods for computation : Brugallé, Chen, Itenberg, Kharlamov, Mikhalkin, Shustin, Solomon, Tukachinsky, Welschinger.

Theorem (G.-Zinger)

The number $N_d^{\mathbb{R}}$ of degree d real rational curves passing through $2d$ conjugate pairs of points in $\mathbb{CP}^3$ is given by

$$N_d^{\mathbb{R}} = - \sum_{\substack{2d_1+d_2=d \\ d_1,d_2 \geq 1}} 4^{d_1-1} d_2 \binom{d-2}{d_2-1} N_{d_1}^{\mathbb{C}} N_{d_2}^{\mathbb{R}},$$

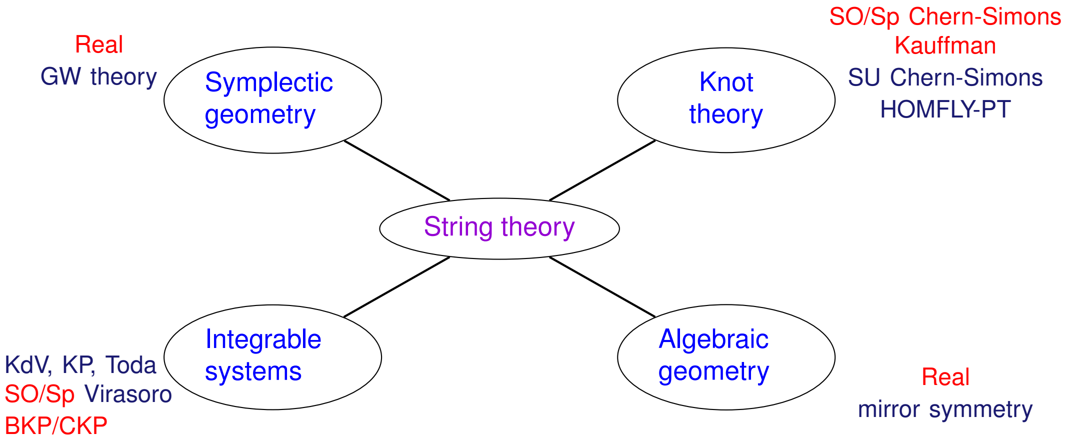
where $N_d^{\mathbb{C}}$ is the classical Gromov-Witten invariant counting rational curves of degree d passing through $2d-1$ points and 2 lines.

Higher genus real GW invariants

Real Gromov-Witten invariants of $\mathbb{CP}^3$ (Niu-Zinger)

d	1	2	3	4	5	6	7	8
$\mathrm{GW}^{\phi}_{0,d}$	1	0	-1	0	5	0	-85	0
$\mathrm{GW}^{\phi}_{1,d}$	0	0	0	-1	0	-4	0	-1000
$\mathrm{GW}^{\phi}_{2,d}$	$\frac{1}{24}$	0	$-\frac{5}{24}$	0	$\frac{15}{8}$	0	$-\frac{1345}{24}$	0
$\mathrm{GW}^{\phi}_{3,d}$	0	0	0	$-\frac{1}{3}$	0	-3	0	$-\frac{2840}{3}$
$\mathrm{GW}^{\phi}_{4,d}$	$\frac{1}{1920}$	0	$-\frac{23}{1152}$	0	$\frac{43}{128}$	0	$-\frac{2475}{128}$	0
$\mathrm{GW}^{\phi}_{5,d}$	0	0	0	$-\frac{19}{360}$	0	$-\frac{16}{15}$	0	$-\frac{1400}{3}$
$\mathrm{E}^{\phi}_{0,d}$	1	0	-1	0	5	0	-85	0
$\mathrm{E}^{\phi}_{1,d}$	0	0	0	-1	0	-4	0	-1000
$\mathrm{E}^{\phi}_{2,d}$	0	0	0	0	0	0	-10	0
$\mathrm{E}^{\phi}_{3,d}$	0	0	0	0	0	-1	0	-280
$\mathrm{E}^{\phi}_{4,d}$	0	0	0	0	0	0	-1	0
$\mathrm{E}^{\phi}_{5,d}$	0	0	0	0	0	0	0	-40

Connections motivated by physics



Local theory

- Let (Σ_g, c) be a genus g symmetric surface, $L \to \Sigma_g$ a complex line bundle, and $(X, \phi) = Tot(L \oplus c^*(\bar{L}))$.
- The real GW theory of such targets is called local theory as they model the neighborhood of a J-holomorphic curve inside a Calabi-Yau 3-fold.
- The associated real moduli spaces are not only orientable but one can choose canonical orientations compatible with degeneration formulas.
- Let $RGW_d(\Sigma, L)$ denote the *connected* generating function and $\mathbb{R}Z_d(\Sigma, L)$, where

$$1 + \sum_d \mathbb{R}Z_d(\Sigma, L)q^d = exp\left(\sum_d RGW_d(\Sigma, L)q^d + \frac{1}{2}GW_d(\Sigma, L)q^{2d}\right)$$

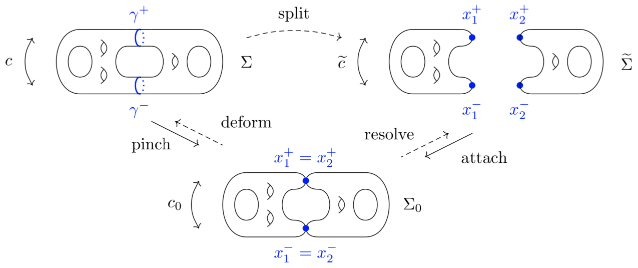
the *disconnected* generating function.

Local theory

The disconnected generating function $\mathbb{R}Z_d(\Sigma, L)$ satisfies a splitting formula (with the canonical choice of orientations)

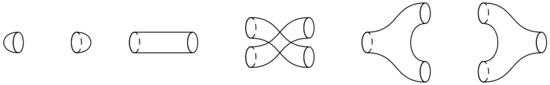
$$\mathbb{R}Z_d(\Sigma, L) = \sum_{\lambda \vdash d} \mathbb{R}Z_d(\Sigma_1, L_1)_\lambda \mathbb{R}Z_d(\Sigma_2, L_2)^\lambda,$$

where $\mathbb{R}Z_d(\Sigma, L)_\lambda$ denotes the restricted count of maps with given ramification profile fixed by a partition λ and $\mathbb{R}Z_d(\Sigma, L)^\lambda = \zeta(\lambda)\mathbb{R}Z_d(\Sigma, L)_\lambda$.

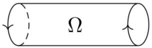


2KCob

- The category **2Cob** :
  - Objects : disjoint unions of oriented circles
  - Morphisms : oriented surfaces with boundary
- The category **2KCob** (forget orientations/orientability) :
  - Objects : disjoint unions of circles
  - Morphisms : surfaces with boundary (possibly non-orientable)
- **2Cob** $\subset$ **2KCob**
- Generators :



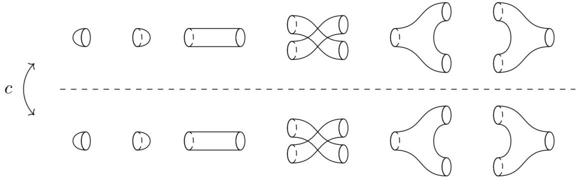
together with the cross-cap (Möbius band) and the involution



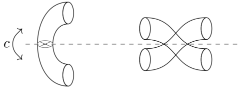
2SymCob

A useful perspective : consider the orientation double cover of both object and morphisms.

- The category **2SymCob** :
  - Objects : disjoint unions of pairs circles $\mathcal{S} = (S^1 \sqcup \overline{S}^1, \epsilon)$
  - Morphisms : oriented surfaces with boundary equipped with an orientation reversing involution c covering ϵ (no fixed points).
- **2KCob** $\cong$ **2SymCob**
- Generators :



together with the cross-cap and the involution



2d KTQFT

- A *2d Klein TQFT with values in a commutative ring R* is a symmetric monoidal functor

$$F : 2\mathbf{KCob} \longrightarrow Rmod$$

(or equivalently $F : 2\mathbf{SymCob} \longrightarrow Rmod$).

- A 2d KTQFT F is equivalent to a commutative Frobenius algebra $H = F(S^1)$ together with two extra structures :
  - an involutive (anti)-automorphism Ω of the Frobenius algebra H, denoted $x \mapsto x^*$.
  - an element $U \in H$ such that

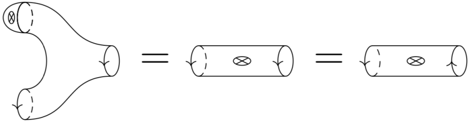
$$(aU)^* = aU \text{ for all } a \in H \text{ and}$$

$$U^2 = m(id \otimes \Omega)(\Delta(1)) = \sum_i \alpha_i \beta_i^* \,, \quad \text{where the coproduct } \Delta(1) = \sum_i \alpha_i \otimes \beta_i$$

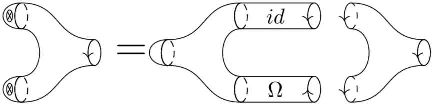
The automorphism Ω corresponds to the involution in $2\mathbf{KCob}$ and the element U to the cross-cap.

Special relations

- the involution acts trivially on the product of the cross-cap with another element



- decomposition of the Klein bottle



Semi-simple theories

- A *semi-simple* (Klein) TQFT is a (Klein) TQFT whose associated Frobenius algebra is semi-simple.
- A semi-simple TQFT is determined by the structure constants λ_ρ, i.e. the coefficients of the co-multiplication $\Delta(v_\rho) = \lambda_\rho v_\rho \otimes v_\rho$ in the idempotent basis v_ρ.
- If F is a semi-simple KTQFT with an idempotent basis $\{v_\rho\}$ then
  - Ω defines an involution on the basis $\Omega(v_\rho) = v_{\rho^*}$;
  - if $U = \sum_\rho U_\rho v_\rho$, then $U_\rho^2 = \lambda_\rho$ if $\rho = \rho^*$ and $U_\rho = 0$ otherwise.

  (There is more than one KTQFT structure with the same underlying TQFT and there is always one in the semi-simple case.)

If Σ is a closed symmetric surface of genus g, considered as a morphism in **2SymCob** from the ground ring to the ground ring, we have

$$F(\Sigma) = \sum_{\rho=\rho^*} U_\rho^{g-1}$$

Motivation : we are interested in the real Gromov-Witten theory of $Tot(L \oplus c^*\overline{L} \longrightarrow \Sigma)$.

- Consider the enlarged category 2**SymCob**^{L} :
  - Objects : disjoint unions of pairs circles $\mathcal{S} = (S^1 \sqcup \bar{S}^1, \epsilon)$
  - Morphisms : oriented surfaces with boundary equipped with an orientation reversing involution c covering ϵ (no fixed points) together with a complex line bundle L trivialized over the boundaries.

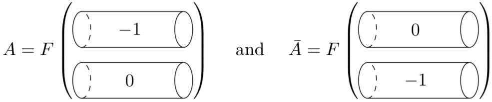
  The Euler class of L determines the bundle up to isomorphism and is the only additional data that we record.

- Generators : same as before together with

- A 2d Klein TQFT on 2**SymCob**^{L} is a symmetric monoidal functor

$$F : 2\mathbf{SymCob}^L \longrightarrow Rmod.$$

The theory is completely determined by the level 0 theory together with the level-decreasing operators



If the theory is semi-simple with idempotent basis $\{v_\rho\}$ then

$$A(v_\rho) = \eta_\rho v_\rho \quad \text{and} \quad \overline{A}(v_\rho) = \overline{\eta}_\rho v_\rho.$$

If Σ is a genus g symmetric surface and $L \longrightarrow \Sigma$ a complex line bundle of Chern class k

$$F(\Sigma|L) = \sum_{\rho=\rho^*} U_\rho^{g-1} \eta_\rho^{-k}.$$

Klein TQFT structure of RGW

Let $R = \mathbb{C}(t)((u))$.

$\mathbf{RGW_d} : 2\mathbf{SymCob}^L \longrightarrow Rmod$

$\mathbf{RGW_d}(S^1 \sqcup \overline{S}^1) = H = \bigoplus_{\alpha \vdash d} Re_\alpha$

To a cobordism $(\Sigma, L) \in 2\mathbf{SymCob}$ from n-copies of $\mathcal{S}$ to m-copies of $\mathcal{S}$, $\mathbf{RGW_d}$ associates R-module homomorphism

$\mathbf{RGW_d}(\Sigma, L) : H^{\otimes n} \longrightarrow H^{\otimes m}$

$$e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_n} \mapsto \sum_{\mu_1,\ldots,\mu_m} \mathbb{R}Z_d(\Sigma_g, L)^{\mu_1,\ldots,\mu_m}_{\lambda_1,\ldots,\lambda_n} e_{\mu_1} \otimes \cdots \otimes e_{\mu_m}.$$

The $\mathbf{RGW_d}$ theory is semi-simple with idempotent basis

$$v_\rho = \frac{\dim \rho}{d!} \sum_\alpha (-t)^{l(\alpha)-d} \chi_\rho(\alpha) e_\alpha$$

- The involution Ω is given by
$$\Omega(e_\alpha) = (-1)^{d-\ell(\alpha)} e_\alpha \quad and \quad \Omega(v_\rho) = v_{\rho'}$$
where ρ' denotes the conjugate representation.
- The cross-cap *U* is given by
$$U = \sum_{\substack{\rho \vdash d \\ \rho=\rho'}} (-1)^{(d-r(\rho))/2}\, t^d \frac{d!}{\dim \rho} v_\rho$$
where $r(\rho)$ is the length of the main diagonal of the Young diagram of ρ.

This uses

– a signed Frobenius-Schur indicator which recognizes self-conjugate representations
$$SFS(\rho) \overset{\text{def}}{=} \frac{1}{d!} \sum_{g \in S_d} \chi_\rho(g^2)(-1)^{s(g)}$$

– Weyl formula for B_n and in particular the identity
$$\sum_{\rho=\rho'} (-1)^{(d+r(\rho))/2} s_\rho(x_1, \ldots, x_n) = \prod_i (1 - x_i) \prod_{i<j} (1 - x_i x_j)$$

Generating series

Let (Σ, c) be a genus g symmetric surface, $L \to \Sigma$ a line bundle of degree $g-1$ and $(X, \phi) = Tot(L \oplus c^*(\bar{L}))$.

Theorem (G.-Ionel)

$$\mathbb{R}Z_d(X) = \sum_{\rho \vdash d, \rho = \rho'} \left(\epsilon(\rho) \prod_{\square \in \rho} 2\sinh\left(\frac{h(\square)u}{2}\right) \right)^{g-1},$$

where $\epsilon(\rho) = (-1)^{\sum_{\alpha\, odd} \frac{\chi^{\rho}(\alpha^2)}{\zeta(\alpha)}} = (-1)^{\frac{|\rho| - r(\rho)}{2}}$.

Coincides with the computation of V. Bouchard, B. Florea, and M. Marinõ of the partition function of SO/Sp Chern-Simons theory on S^3.

Real Gopakumar-Vafa formula

Let (Σ, c) a genus g symmetric surface, $L \to \Sigma$ a line bundle of degree $g-1$ and $(X, \phi) = Tot(L \oplus c^*(\bar{L}))$.

Theorem (G.-Ionel)

$$\sum_d RGW_{d,h} q^d u^{h-1} = \sum_{d \neq 0, h \geq 0} n^{\mathbb{R}}_{d,h}(g) \sum_{k>0, k\ odd} \frac{1}{k}\left(2\sinh\left(\frac{ku}{2}\right)\right)^{h-1} q^{kd},$$

where $n^{\mathbb{R}}_{d,h}(g) \in \mathbb{Z}$.

This is the local version of the real GV formula proposed by J. Walcher.

Thank you!