Arithmetic Dynamics: A Survey

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Number Theory (3) and Dynamical Systems (9)

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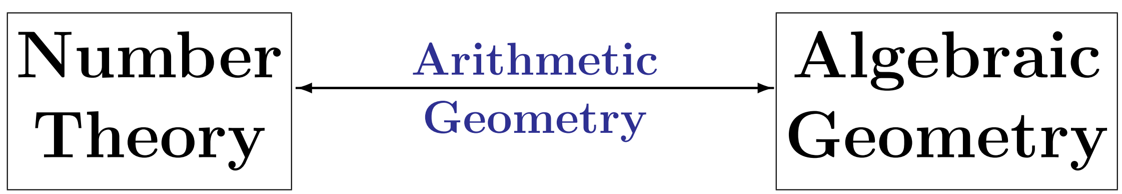

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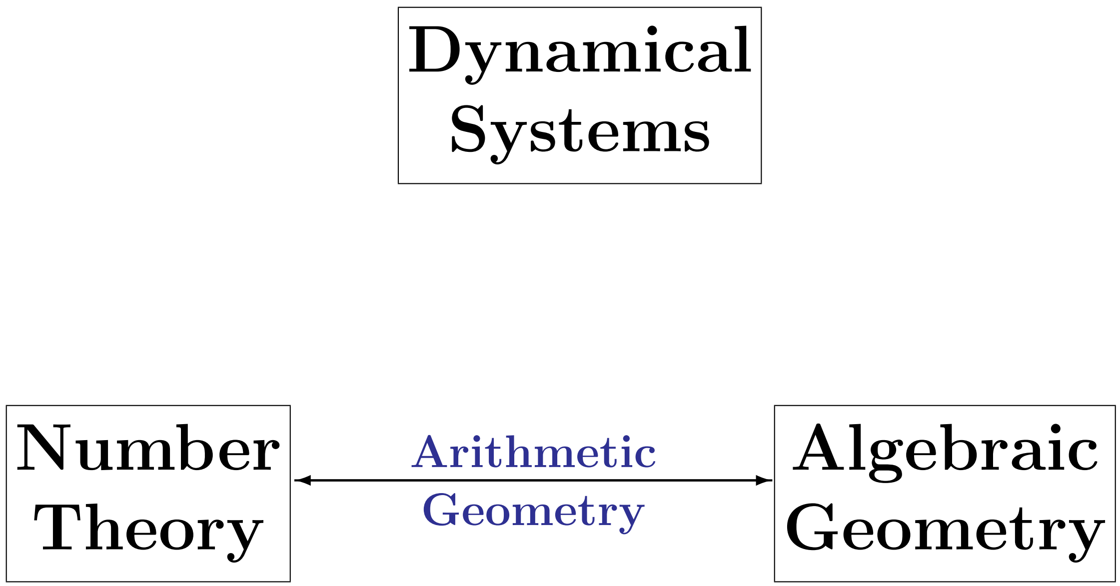

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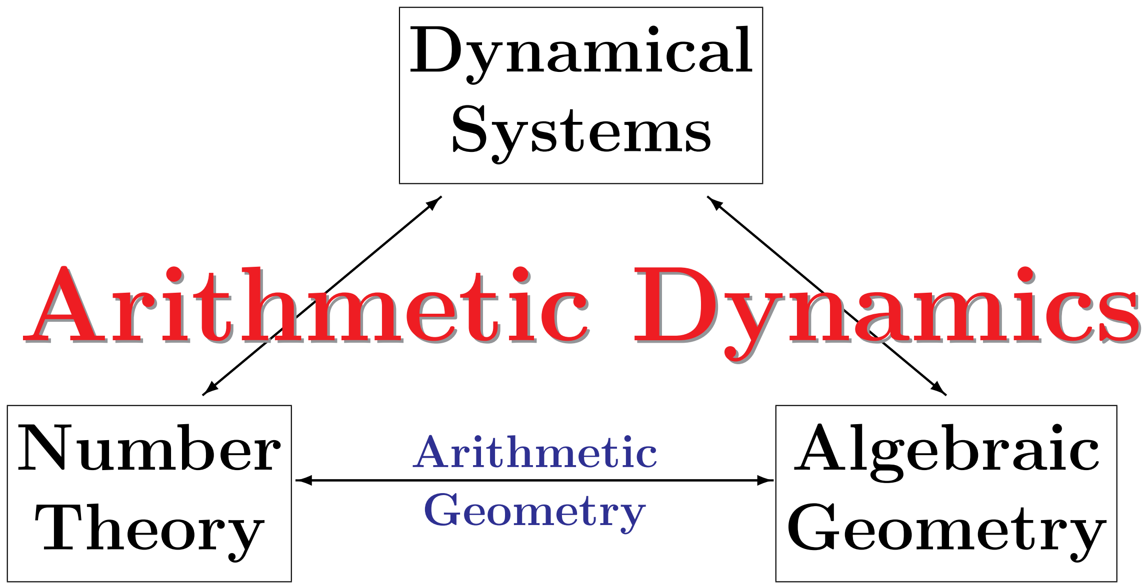

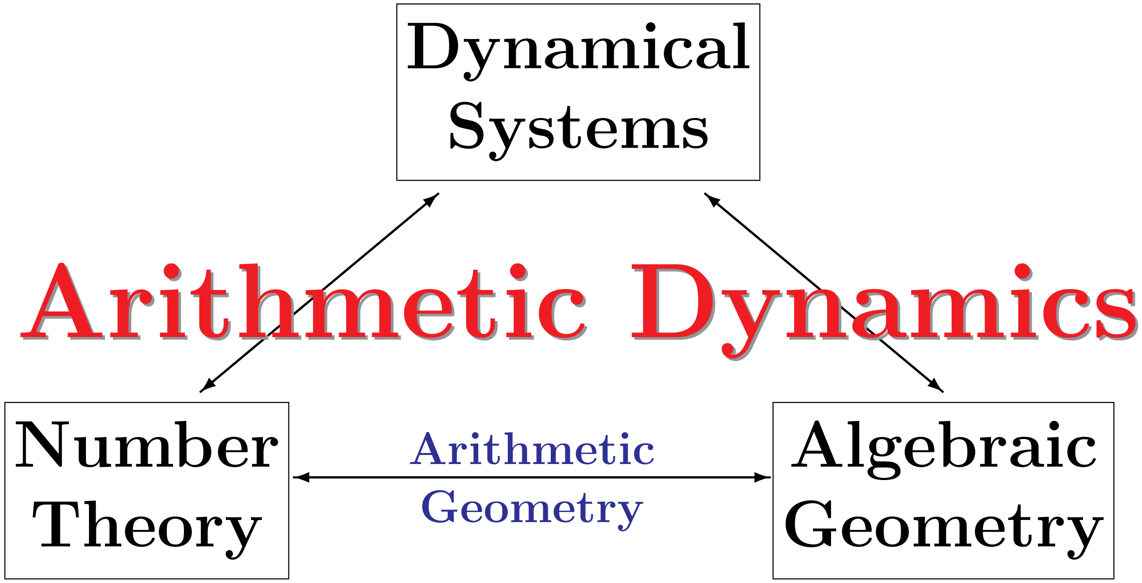

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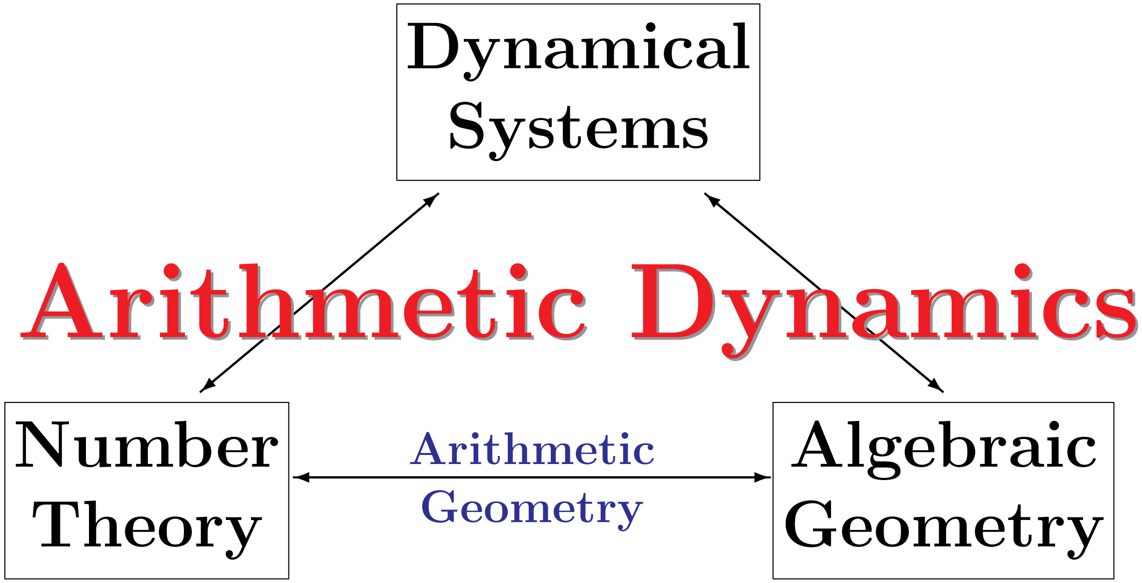

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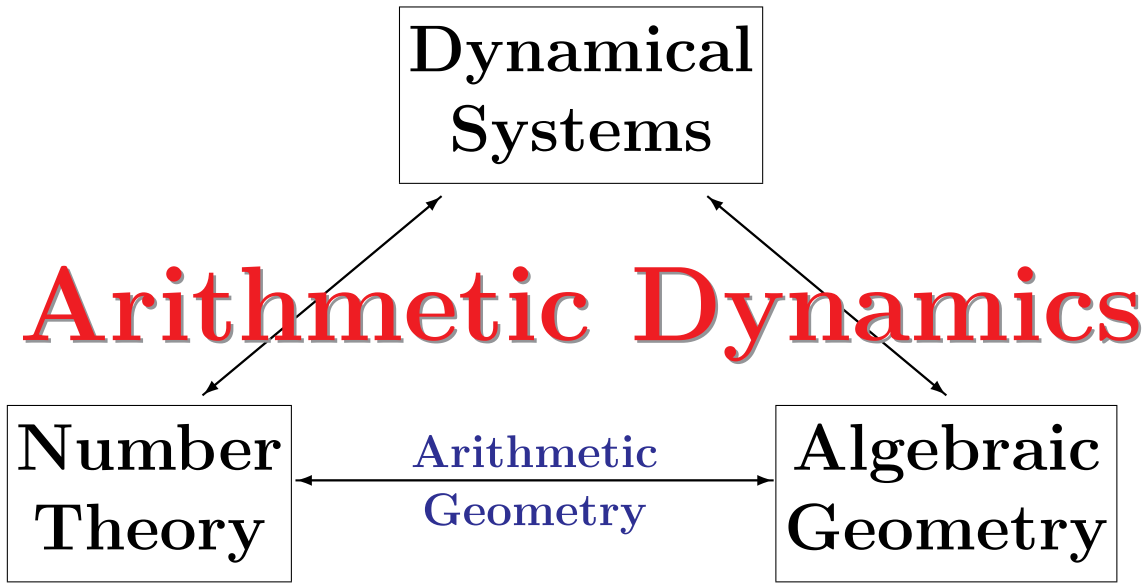

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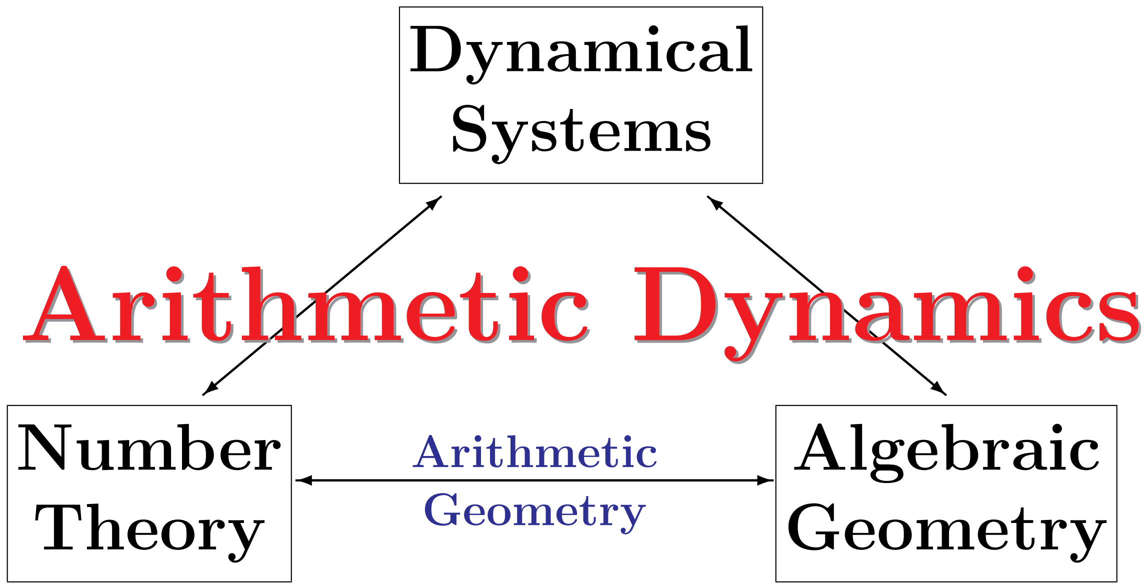

- **Number Theory**: Study properties of integers and rational numbers
- **Algebraic Geometry**: Study solutions to systems of polynomial equations
- **(Discrete) Dynamical Systems**: Study orbits for iteration of functions
- **Arithmetic Dynamics**: Study number theoretic properties of orbits of rational numbers for iteration of polynomial or rational functions

Outline of this Talk

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My primary focus is on dynamical analogues of famous theorems and conjectures in arithmetic geometry, centered around five major topics that have helped drive the development of arithmetic dynamics over the past few decades:

- Topic #1: Dynamical Uniform Boundedness
- Topic #2: Dynamical and Arithmetic Complexity
- Topic #3: Dynamical Moduli Spaces
- Topic #4: Dynamical Unlikely Intersections
- Topic #5: Dynatomic and Arboreal Representations

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Some Dynamical Definitions

- A **dynamical system** on a set X is a function

$$f : X \longrightarrow X.$$

- The **iterates of f** are denoted by

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ copies of } f}.$$

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- The **(forward) f-orbit of $x \in X$** is
$$\mathcal{O}_f(x) = \{f^n(x) : n \geq 0\}.$$
- An element $x \in X$ is **f-periodic** if
$$f^n(x) = x \quad \text{for some } n \geq 1.$$
The smallest such n is the **exact period**.
- An element $x \in X$ is **f-preperiodic** if
$$f^{n+m}(x) = f^m(x) \quad \text{for some } n \geq 1 \text{ and } m \geq 0.$$
Equivalently x is preperiodic if $\mathcal{O}_f(x)$ is finite.

A Dictionary for Arithmetic Dynamics

Arithmetic Geometry	**Dynamical Systems**
rational and integral points on varieties	rational and integral points in orbits
torsion points on abelian varieties	periodic and preperiodic points of rational maps
abelian varieties with complex multiplication	post-critically finite rational maps on $\mathbb{P}^1$

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- The first analogy is very natural.
- The second analogy is, too, since if G is a group, then

$$\left(\begin{array}{l} g \in G \text{ has} \\ \text{finite order} \end{array}\right) \iff \left(\begin{array}{l} g \text{ is preperiodic for} \\ \text{the map } x \to x^2. \end{array}\right)$$

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- The third analogy is a bit more tenuous. It is based on the fact that each corresponds to a countable collection of special algebraic points in the associated moduli space. Work of Baker–DeMarco (2011), Ghoica–Krieger–Nguyen–Ye (2017), Favre–Gauthier (2020) and others provides some justification for the third analogy.

Topic #1: Dynamical Uniform Boundedness

A famous theorem (extended by Kamienny and Merel):

Theorem. (Mazur 1970s) Let $E/\mathbb{Q}$ be an elliptic curve. Then

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The dictionary on the previous slide says:

$$\text{(torsion points)} \longleftrightarrow \text{(preperiodic points)}.$$

This leads to...

The Dynamical Uniform Boundedness Conjecture

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$$f : \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

be a morphism (holomorphic map) of degree d that is defined over $\mathbb{Q}$. Then

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The DUBC remains far out of reach.

- The DUBC for $N = 1$ and $d = 4$ easily implies a version of Mazur's theorem.
- Fakhruddin (2001) has shown that the DUBC for all $N \ge 1$ implies uniform boundedness of torsion on abelian varieties of every dimension.

Uniform Boundedness for Quadratic Polynomials

Since the general DUBC seems inaccessible, we consider a small family of functions. And what better family than the ubiquitous quadratic polynomials whose study motivates so much research in complex dynamics:

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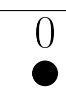
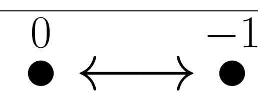
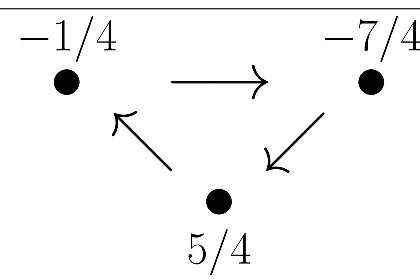
$n = 1$	x^2	$c = 0$	$b = 0$	
$n = 2$	$x^2 - 1$	$c = -1$	$b = 0$	
$n = 3$	$x^2 + \frac{29}{16}$	$c = \frac{29}{16}$	$b = -\frac{1}{4}$	

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- $\boldsymbol{n > n_0}$: No.* (Looper 2010s)

  * assuming an *abcd*...-type conjecture

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Idea of Proof of DUBC for $x^2 + c$:

- There is a **dynamical modular curve** $X_1^{\mathrm{dyn}}(n)$ that classifies pairs $(b, c) \in \mathbb{A}^2$ such that b has exact period n for $x^2 + c$.

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- A key tool in Mazur's proof for elliptic curves is the existence of Hecke correspondences on $X_1^{\text{ell}}(n)$. Unfortunately, there do not appear to be analogous correspondences on $X_1^{\text{dyn}}(n)$.

Topic #2: Dynamical and Arithmetic Complexity

We take an informal approach to complexity by defining the **complexity** of an object $\mathcal{O}$ to be

$$\begin{aligned} h(\mathcal{O}) &= \text{complexity of } \mathcal{O} \\ &\asymp \#\text{ of basic units needed to describe } \mathcal{O}. \end{aligned}$$

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Examples:

$$h\left(\text{rational number } \frac{p}{q}\right) = \log \max\bigl\{|p|, |q|\bigr\}.$$

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Goal: Study how fast complexity grows for iterates and in orbits.

Dynamical Complexity of Rational Maps

A **rational map of degree *d*** is a function

$$f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N, \quad f = [f_0, \ldots, f_N],$$

where

$$f_0, \ldots, f_N \in K[x_0, \ldots, x_N]$$

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Example: The map

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad f = [x_1x_2, x_0x_1, x_2^2]$$

satisfies

$$\deg(f^n) = \text{Fibonacci}_{n+2}.$$

The Dynamical Degree of a Rational Map

The **dynamical degree** of a rational map

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Example: For the example on the previous slide,

$$\delta_f = \lim_{n\to\infty} (\text{Fibonacci}_{n+2})^{1/n} = \frac{1+\sqrt{5}}{2}.$$

Properties of Dynamical Degrees

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Theorem. (Bell–Diller–Jonsson–Krieger, 2019, 2021): For all $N \geq 3$, there exist birational automorphisms

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General Definition: Let X be a projective variety. The **dynamical degree** of a rational map $f : X \dashrightarrow X$ is

$$\delta_f = \lim_{n\to\infty} \Bigl(\deg_{\mathcal{L}}(f^n)\Bigr)^{1/n}.$$

(The degree is relative to any ample line bundle $\mathcal{L}$ on X.)

Number Theoretic Complexity

We can define the complexity (**height**) of a point $P \in \mathbb{P}^N(\mathbb{Q})$ by writing

$$P = [a_0, \ldots, a_N] \quad \text{with} \quad \begin{cases} a_0, \ldots, a_N \in \mathbb{Z}, \\ \gcd(a_0, \ldots, a_N) = 1, \end{cases}$$

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More generally for any algebraic variety X defined over a number field K we let

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Question: Let $f : X \to X$. How fast does $h\bigl(f^n(P)\bigr)$ grow?

Arithmetic Degree

For a rational map

$$f : X \dashrightarrow X$$

we must avoid points where f is not defined, so we let

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The **arithmetic degree of the f-orbit of $P \in X_f$** is

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In any case we can define an **upper arithmetic degree** by

$$\overline{\alpha}_f(P) = \limsup_{n\to\infty} \max\left\{h\bigl(f^n(P)\bigr),\, 1\right\}^{1/n}.$$

Comparing Dynamical and Arithmetic Degrees

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Moral of the Theorem: The number theoretic complexity of points in an orbit never exceeds the dynamical/geometric complexity of the map itself.

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Moral of the Theorem: The number theoretic complexity of points in an orbit never exceeds the dynamical/geometric complexity of the map itself.

Question: When are they equal?

Comparing Dynamical and Arithmetic Degrees

Conjecture. (Kawaguchi–Silverman)

$$\mathcal{O}_f(P) \text{ Zariski dense in } X \implies \overline{\alpha}_f(P) = \delta_f.$$

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Moral of the Conjecture: An orbit with maximal dynamical/geometric complexity also has maximal number theoretic complexity.

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Work on this conjecture has used tools ranging from linear-forms-in-logarithms to canonical heights for nef divisors to the minimal model program in algebraic geometry. Known cases (by many authors) include:

- The composition of a homomorphism and a translation on a semi-abelian variety.
- Endomorphisms of (not necessarily smooth) projective surfaces.
- $\mathbb{P}^N$-extensions of regular affine automorphisms of $\mathbb{A}^N$.
- Non-invertible endomorphisms of smooth projective 3-folds of Kodaira dimenson 0.
- "Int-amplified" endomorphisms of smooth projective 3-folds.

Sketch of a Proof

Since every talk is supposed to include a proof, I'll sketch the proof of the conjecture for abelian varieties.

Theorem. (Kawaguchi–Silverman) Let

$$\begin{aligned} K/\mathbb{Q} &\quad \text{a number field,} \\ A/K &\quad \text{an abelian variety,} \\ f : A \to A &\quad \text{a non-zero isogeny,} \\ P \in A(K) &\quad \text{an algebraic point.} \end{aligned}$$

Then

$$\overline{\alpha}_f(P) < \delta_f \implies \mathcal{O}_f(P) \text{ is not Zariski dense in } A.$$

Sketch of a Proof (continued)

1. Consider the action of f^* on $\mathrm{NS}(A) \otimes \mathbb{R}$. Let D_f be an eigendivisor class with maximal eigenvalue. One shows that the eigenvalue is δ_f.
2. The assumption $\overline{\alpha}_f(P) < \delta_f$ implies $\hat{h}_{D_f}(P) = 0$.
3. The divisor D_f is nef, i.e., it is on the boundary of the ample cone. The heart of the proof uses:

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Lemma. (K-S) For all non-zero nef divisor classes $D \in \mathrm{NS}(A) \otimes \mathbb{R}$, there exists an abelian subvariety $B_D \subsetneq A$ such that

$$\hat{h}_D(P) = 0 \quad \Longleftrightarrow \quad P \in B_D + A_{\mathrm{tors}}.$$

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4. One shows further that $f(B_{D_f}) = B_{D_f}$. Hence

$$\overline{\alpha}_f(P) < \delta_f \implies \mathcal{O}_f(P) \in \underbrace{B_{D_f} + A_{\mathrm{tors}}(K)}_{\text{This is not Zariski dense in } A.}.$$

Topic #3: Dynamical Moduli Space

The goal is to construct a space whose points classify dynamical systems. We start with

$$\mathrm{End}_d^N = \{\text{degree } d \text{ morphisms } f : \mathbb{P}^N \longrightarrow \mathbb{P}^N\}.$$

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A map f is specified by $N+1$ homogeneous degree d polynomials in $N+1$ variables,

$$f = [f_0, \ldots, f_N].$$

Using the coefficients of $f_0, \ldots, f_N$ to define a point, we may identify End_d^N with an affine subvariety of $\mathbb{P}^\nu$.

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We define an action

$$f^\phi = \phi^{-1} \circ f \circ \phi \quad \text{for } \phi \in \mathrm{Aut}(\mathbb{P}^N) = \mathrm{PGL}_{N+1}.$$

This action commutes with iteration,

$$(f^\phi)^n = (f^n)^\phi,$$

so f and f^ϕ have the same dynamics.

The Dynamical Moduli Space $\mathcal{M}_d^N$

Definition. The **moduli space of degree d dynamical systems on $\mathbb{P}^N$** is the quotient space

$$\mathcal{M}_d^N = \mathrm{End}_d^N \,/\, \mathrm{PGL}_{N+1}\,.$$

A point of $\mathcal{M}_d^N$ corresponds to a degree d dynamical system on $\mathbb{P}^N$ up to change of coordinates.

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We would like the abstract quotient $\mathcal{M}_d^N$ to have additional structure.

Theorem. Let $N \geq 1$ and $d \geq 2$.

(a) (Milnor '93) $\mathcal{M}_d^N(\mathbb{C})$ is an orbifold over $\mathbb{C}$.

(b) (Levy '11, Petsche–Szpiro–Tepper '09, Silverman '98) $\mathcal{M}_d^N$ is a GIT geometric quotient scheme over $\mathbb{Z}$. In particular, $\mathcal{M}_d^N$ exists as an algebraic variety over fields of arbitrary characteristic.

The Structure of $\mathcal{M}_d^N$

For $N = 1$, i.e., for maps of $\mathbb{P}^1$, we have some knowledge about $\mathcal{M}_d^N$.

Theorem. (a) (Milnor '93, Silverman '98) There is an isomorphism

$$\mathcal{M}_2^1 \cong \mathbb{A}^2,$$

and the natural compactification of $\mathcal{M}_2^1$ is isomorphic to $\mathbb{P}^2$.

(b) (Levy '11) There is a birational isomorphism

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Question. Is $\mathcal{M}_d^N$ birationally isomorphic to projective space for all $d \geq 2$ and all $N \geq 1$?

Multiplier Systems

Let $f : \mathbb{P}^N \to \mathbb{P}^N$ and let $P \in \mathbb{P}^N$ be a point of exact period n. The associated **multipliers** are the eigenvalues of the induced map

$$(f^n)^* : \mathcal{T}_P \longrightarrow \mathcal{T}_P.$$

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The sets of multipliers of f and f^ϕ are the same. This allows us to define a **multiplier map**

$$\mu_{N,d,n} : \mathcal{M}_d^N \longrightarrow \mathbb{A}^\nu$$

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Theorem. (McMullen '87) Fix $d \geq 2$. Then for all sufficiently large n, the multiplier map

$$\mu_{1,d,n} : \mathcal{M}_d^1 \longrightarrow \mathbb{A}^\nu$$

is finite-to-one (except if d is a square).

Dynamical Moduli Spaces with Level Structure

Just as algebraic geometers study pairs (A, P) consisting of an abelian variety A and a torsion point P of exact order n, we can consider dynamical moduli spaces with level structure:

$$\mathcal{M}_d^N[n] = \frac{\bigl\{(f, P) \in \operatorname{End}_d^N \times \mathbb{P}^N : P \text{ has exact } f\text{-period } n\bigr\}}{\operatorname{PGL}_{N+1}}.$$

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Example. The space $\mathcal{M}_2^1[n]$ classifies degree 2 rational maps $\mathbb{P}^1 \to \mathbb{P}^1$ with a marked point of period n. It is a surface with a ramified covering map

$$\mathcal{M}_2^1[n] \longrightarrow \mathcal{M}_2^1 \cong \mathbb{A}^2.$$

The dynamical modular curve $X_1^{\mathrm{dyn}}(n)$ mentioned earlier, which classifies degree 2 polynomials, sits as a subvariety of $\mathcal{M}_2^1[n]$.

The Structure of Dynamical Moduli Spaces

We start with a conjecture, inspired by Tai's theorem on the moduli spaces of abelian varieties.

Conjecture. Fix $N \ge 1$ and $d \ge 2$.

(a) The moduli space $\mathcal{M}_d^N[n]$ is irreducible for all $n \ge 1$.

(b) The moduli space $\mathcal{M}_d^N[n]$ is of general type for all $n \ge C(N, d)$.

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Our knowledge to date is fragmentary.

Theorem.

(a) (Manes '09) $\mathcal{M}_2^1[n]$ is irreducible for all $n \ge 1$.

(b) (Blanc–Canci–Elkies '15) $\mathcal{M}_2^1[6]$ is surface of general type.

(c) (Bousch '92) $X_1^{\text{dyn}}(n)(\mathbb{C})$ is irreducible for all $n \ge 1$.

(d) (Morton '96) $\text{genus}\bigl(X_1^{\text{dyn}}(n)\bigr) \to \infty$ as $n \to \infty$.

(e) (Doyle–Poonen '17) $\text{gonality}\bigl(X_1^{\text{dyn}}(n)\bigr) \to \infty$ as $n \to \infty$.

This concludes our brief tour of some topics in arithmetic dynamics.

I want to thank the IMU for inviting me to speak at ICM 2022, and to thank you for your attention.

Arithmetic Dynamics:
A Survey

Joseph H. Silverman

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International Congress of Mathematicians 2022
Number Theory (3) and Dynamical Systems (9)

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