Mean curvature and variational theory

Xin Zhou

Cornell University

-

Geometry Section, ICM 2022

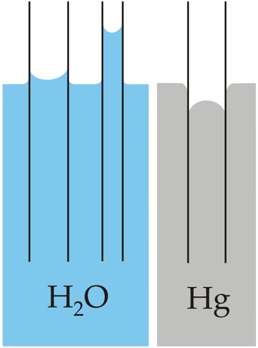
-

July 13, 2022

Capillary phenomenon

When putting a thin tube into a liquid, the liquid will be pulled up/pushed down. This is usually called the Capillary phenomenon/action.

Liquid molecules near the wall are pulled toward/away the wall, and cohesive forces carry the remaining liquid molecules with them toward/away the wall. The pressure at the top of the liquid column is compensated by the curvature of the surface.

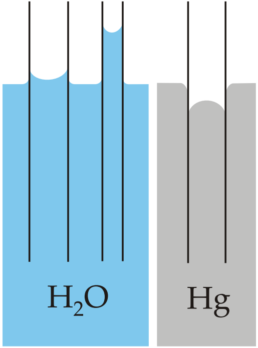


(From: Wikipedia)

Capillary phenomenon

When putting a thin tube into a liquid, the liquid will be pulled up/pushed down. This is usually called the Capillary phenomenon/action.

Liquid molecules near the wall are pulled toward/away the wall, and cohesive forces carry the remaining liquid molecules with them toward/away the wall. The pressure at the top of the liquid column is compensated by the curvature of the surface.



(From: Wikipedia)

Equations governing capillary action

In 1805, T. Young introduced the notion of mean curvature $\boldsymbol{H}$ of a surface, and wrote down the equation satisfied by capillary surface:

$$\Delta p = 2\sigma \boldsymbol{H},$$

where Δp is the pressure change, and σ is the surface tension.

Around the same time P.-S. Laplace derived the formula for mean curvature of the capillary surface, and the equation for the height function $u : \Omega \to \mathbb{R}$:

$$\boldsymbol{H} \equiv \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \kappa u + \lambda. \tag{1}$$

Equations governing capillary action

In 1805, T. Young introduced the notion of mean curvature $\boldsymbol{H}$ of a surface, and wrote down the equation satisfied by capillary surface:

$$\Delta p = 2\sigma \boldsymbol{H},$$

where Δp is the pressure change, and σ is the surface tension.

Around the same time P.-S. Laplace derived the formula for mean curvature of the capillary surface, and the equation for the height function $u : \Omega \to \mathbb{R}$:

$$\boldsymbol{H} \equiv \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \kappa u + \lambda. \tag{1}$$

Prescribing mean curvature equations

Denote $\Sigma_u = \{(x, u(x)) : x \in \Omega\} \subset \Omega \times \mathbb{R}$.

Equation (1) belongs to a large class of prescribing mean curvature (PMC) equations:

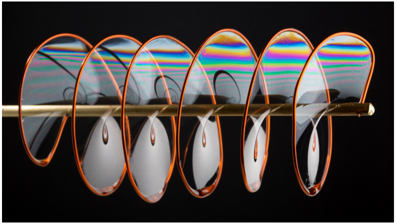
$$\boldsymbol{H}(\Sigma_u) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = f(u).$$

- When $f \equiv 0$, this reduces to the minimal surface equation;
- When f is a non-zero constant, this reduces to the constant mean curvature (CMC) equation.

Minimal and CMC surfaces

Minimal surfaces:

Put a closed wire into soap liquid. It will bound a soap film. The soap film minimizes area due to the surface tension, and hence is a model for minimal surface.



(From: Quanta Magazine)

CMC surfaces:

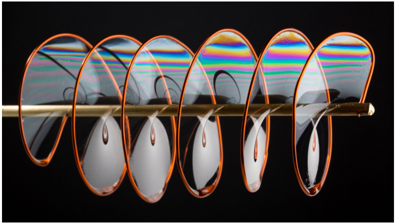
Blow out a soap bubble. The bubble surface minimizes area while keeping the enclosed volume fixed. This is a model for CMC surface.

(From: Wikipedia)

Minimal and CMC surfaces

Minimal surfaces:

Put a closed wire into soap liquid. It will bound a soap film. The soap film minimizes area due to the surface tension, and hence is a model for minimal surface.



(From: Quanta Magazine)

CMC surfaces:

Blow out a soap bubble. The bubble surface minimizes area while keeping the enclosed volume fixed. This is a model for CMC surface.

(From: Wikipedia)

Mean curvature and Area

Let Σ^2 be a surface in $\mathbb{R}^3$. The *mean curvature* $\boldsymbol{H}$ measures how the area of Σ changes.

The first variation of Area of Σ along a vector field X is given by

$$\delta\,\mathrm{Area}_\Sigma(X) = \int_\Sigma \boldsymbol{H}\langle \boldsymbol{n}, X\rangle,$$

where $\boldsymbol{n}$ is a unit normal of Σ.

Σ is a *minimal surface* if it is a critical point of Area; that is $\delta\,\mathrm{Area}_\Sigma(X) = 0$; that is:

$$\boldsymbol{H}_\Sigma \equiv 0.$$

Σ is a *CMC surface* if it is a critical point of Area among volume-preserving variations of Ω, i.e. $\int_\Sigma \langle \boldsymbol{n}, X\rangle = 0$; that is,

$$\boldsymbol{H}_\Sigma \equiv \text{const}.$$

Mean curvature and Area

Let Σ^2 be a surface in $\mathbb{R}^3$. The *mean curvature* $\boldsymbol{H}$ measures how the area of Σ changes.

The first variation of Area of Σ along a vector field X is given by

$$\delta\,\mathrm{Area}_\Sigma(X) = \int_\Sigma \boldsymbol{H}\langle \boldsymbol{n}, X\rangle,$$

where $\boldsymbol{n}$ is a unit normal of Σ.

Σ is a *minimal surface* if it is a critical point of Area; that is $\delta\,\mathrm{Area}_\Sigma(X) = 0$; that is:

$$\boldsymbol{H}_\Sigma \equiv 0.$$

Σ is a *CMC surface* if it is a critical point of Area among volume-preserving variations of Ω, i.e. $\int_\Sigma \langle \boldsymbol{n}, X\rangle = 0$; that is,

$$\boldsymbol{H}_\Sigma \equiv \text{const}.$$

Mean curvature and Area

Let Σ^2 be a surface in $\mathbb{R}^3$. The *mean curvature* $\boldsymbol{H}$ measures how the area of Σ changes.

The first variation of Area of Σ along a vector field X is given by

$$\delta \operatorname{Area}_\Sigma(X) = \int_\Sigma \boldsymbol{H}\langle \boldsymbol{n}, X\rangle,$$

where $\boldsymbol{n}$ is a unit normal of Σ.

Σ is a *minimal surface* if it is a critical point of Area; that is $\delta \operatorname{Area}_\Sigma(X) = 0$; that is:

$$\boldsymbol{H}_\Sigma \equiv 0.$$

Σ is a *CMC surface* if it is a critical point of Area among volume-preserving variations of Ω, i.e. $\int_\Sigma \langle \boldsymbol{n}, X\rangle = 0$; that is,

$$\boldsymbol{H}_\Sigma \equiv \text{const}.$$

Interests and applications

Minimal/CMC surfaces

1. are models for soap films, soap bubbles and interface phenomenon;
2. act as a driving force for the development of modern PDE theory and calculus of variations;
3. are important tools in the study of geometry and topology.
4. General Relativity: boundary of event horizons, definition of the center of mass of an isolated gravitational system.

The classical Plateau's problem

In 1760, Lagrange raised the problem of finding a surface with a given boundary which minimizes area. In the 19th century Plateau did the famous physical experiments using wires and soap films.

Plateau Problem

Given a simple closed curve $\Gamma \subset \mathbb{R}^3$, consider all parametrized disks spanning Γ: $v : D^2 \to \mathbb{R}^3$ with $v : \partial D^2 \to \Gamma$ a monotone map.

Can we find an area minimizer among such maps?

Douglas, Rado 1930s: solved this by minimizing the Dirichlet energy

$$E(v) = \frac{1}{2}\int_D |\nabla v|^2 dxdy.$$

Morrey 1948: generalized this to Riemannian manifolds.

The classical Plateau's problem

In 1760, Lagrange raised the problem of finding a surface with a given boundary which minimizes area. In the 19th century Plateau did the famous physical experiments using wires and soap films.

Plateau Problem

Given a simple closed curve $\Gamma \subset \mathbb{R}^3$, consider all parametrized disks spanning Γ: $v : D^2 \to \mathbb{R}^3$ with $v : \partial D^2 \to \Gamma$ a monotone map.

Can we find an area minimizer among such maps?

Douglas, Rado 1930s: solved this by minimizing the Dirichlet energy

$$E(v) = \frac{1}{2} \int_D |\nabla v|^2 dxdy.$$

Morrey 1948: generalized this to Riemannian manifolds.

Higher dimensions and co-dimensions

To prove the existence of a minimizer Σ^n spanning a given boundary Γ^{n-1} in $\mathbb{R}^{n+k}$, De Giorgi, Federer, Fleming, etc. developed Geometric Measure Theory.

In co-dimensional 1 cases, i.e. $k = 1$, by combining works of De Giorgi, Federer, Fleming, Almgren, Simons, an area minimizer Σ^n is smoothly embedded outside a singular set of codim-7.

When $k > 1$, an area minimizer Σ^n has a codim-2 singular set by deep works of Almgren 1990s, and De Lellis-Spadaro 2013.

Higher dimensions and co-dimensions

To prove the existence of a minimizer Σ^n spanning a given boundary Γ^{n-1} in $\mathbb{R}^{n+k}$, De Giorgi, Federer, Fleming, etc. developed Geometric Measure Theory.

In co-dimensional 1 cases, i.e. $k = 1$, by combining works of De Giorgi, Federer, Fleming, Almgren, Simons, an area minimizer Σ^n is smoothly embedded outside a singular set of codim-7.

When $k > 1$, an area minimizer Σ^n has a codim-2 singular set by deep works of Almgren 1990s, and De Lellis-Spadaro 2013.

Higher dimensions and co-dimensions

To prove the existence of a minimizer Σ^n spanning a given boundary Γ^{n-1} in $\mathbb{R}^{n+k}$, De Giorgi, Federer, Fleming, etc. developed Geometric Measure Theory.

In co-dimensional 1 cases, i.e. $k = 1$, by combining works of De Giorgi, Federer, Fleming, Almgren, Simons, an area minimizer Σ^n is smoothly embedded outside a singular set of codim-7.

When $k > 1$, an area minimizer Σ^n has a codim-2 singular set by deep works of Almgren 1990s, and De Lellis-Spadaro 2013.

CMC Plateau's problem

Given a simple closed curve $\gamma \subset \Omega \subset \mathbb{R}^3$, and $0 < H \in \mathbb{R}$, find a mapping $\phi : D \to \Omega$ spanning γ which has prescribed mean curvature H.

- Heinz 54, Hildebrandt 70: found minimizers of

$$E_H(v) = \frac{1}{2}\int_D |\nabla v|^2 dxdy - \frac{H}{3}\int_D v \cdot (v_x \wedge v_y) dxdy,$$

when H satisfies certain natural upper bound.

Closed solutions – minimal case

Homology: If $H_n(M^{n+k}, \mathbb{Z} \text{ or } \mathbb{Z}_2) \neq 0$, we can use integral current theory to obtain a minimizer Σ_0 in a homology class. When $k = 1$, Σ_0 is smoothly embedded outside a codom-7 singular set.

Homotopy: If M^n contains an incompressible surface Σ^2, Schoen-Yau 79 and Sacks-Uhlenbeck 81 proved the existence of a branched immersed minimizer Σ_0.

Isotopy: In dimension 3, if M^3 contains an incompressible embedded surface Σ^2, Meeks-Simon-Yau 82 obtained an embedded minimizer Σ_0 isotopic to Σ.

Closed solutions – minimal case

Homology: If $H_n(M^{n+k}, \mathbb{Z} \text{ or } \mathbb{Z}_2) \neq 0$, we can use integral current theory to obtain a minimizer Σ_0 in a homology class. When $k = 1$, Σ_0 is smoothly embedded outside a codom-7 singular set.

Homotopy: If M^n contains an incompressible surface Σ^2, Schoen-Yau 79 and Sacks-Uhlenbeck 81 proved the existence of a branched immersed minimizer Σ_0.

Isotopy: In dimension 3, if M^3 contains an incompressible embedded surface Σ^2, Meeks-Simon-Yau 82 obtained an embedded minimizer Σ_0 isotopic to Σ.

Closed solutions – minimal case

Homology: If $H_n(M^{n+k}, \mathbb{Z} \text{ or } \mathbb{Z}_2) \neq 0$, we can use integral current theory to obtain a minimizer Σ_0 in a homology class. When $k = 1$, Σ_0 is smoothly embedded outside a codom-7 singular set.

Homotopy: If M^n contains an incompressible surface Σ^2, Schoen-Yau 79 and Sacks-Uhlenbeck 81 proved the existence of a branched immersed minimizer Σ_0.

Isotopy: In dimension 3, if M^3 contains an incompressible embedded surface Σ^2, Meeks-Simon-Yau 82 obtained an embedded minimizer Σ_0 isotopic to Σ.

Closed solutions – CMC case

Isoperimetric problem:

- Boundaries of isoperimetric domains are closed CMC hypersurfaces (Almgren 76, Morgan 03).

Perturbation method:

- One can obtain foliations of CMC hypersurfaces by perturbing tubular neighborhoods of a point, or a minimal submanifold in nondegenerate scenario (Ye 91, Mahmoudi-Mazzeo-Pacard 06, etc.).

Gluing method:

- Many complete and compact CMC surfaces were constructed in $\mathbb{R}^d$, $d \geq 3$ by gluing methods (Kapouleas 90, Breiner-Kapouleas 17).

Closed solutions – CMC case

Isoperimetric problem:

- Boundaries of isoperimetric domains are closed CMC hypersurfaces (Almgren 76, Morgan 03).

Perturbation method:

- One can obtain foliations of CMC hypersurfaces by perturbing tubular neighborhoods of a point, or a minimal submanifold in nondegenerate scenario (Ye 91, Mahmoudi-Mazzeo-Pacard 06, etc.).

Gluing method:

- Many complete and compact CMC surfaces were constructed in $\mathbb{R}^d$, $d \geq 3$ by gluing methods (Kapouleas 90, Breiner-Kapouleas 17).

Closed solutions – CMC case

Isoperimetric problem:

- Boundaries of isoperimetric domains are closed CMC hypersurfaces (Almgren 76, Morgan 03).

Perturbation method:

- One can obtain foliations of CMC hypersurfaces by perturbing tubular neighborhoods of a point, or a minimal submanifold in nondegenerate scenario (Ye 91, Mahmoudi-Mazzeo-Pacard 06, etc.).

Gluing method:

- Many complete and compact CMC surfaces were constructed in $\mathbb{R}^d$, $d \geq 3$ by gluing methods (Kapouleas 90, Breiner-Kapouleas 17).

General existence questions

Question 1:

Can we find a closed minimal hypersurface Σ^n in an arbitrary closed manifold M^{n+1} (without any assumption on homology, homotopy, or isotopy)?

Question 2:

Can we find a closed CMC hypersurface Σ^n with an arbitrary prescribed curvature $c \in \mathbb{R}$ in an arbitrary closed manifold M^{n+1}?

The answers to both questions rely on a scheme to find saddle point solutions of some variational problems – the min-max method.

General existence questions

Question 1:

Can we find a closed minimal hypersurface Σ^n in an arbitrary closed manifold M^{n+1} (without any assumption on homology, homotopy, or isotopy)?

Question 2:

Can we find a closed CMC hypersurface Σ^n with an arbitrary prescribed curvature $c \in \mathbb{R}$ in an arbitrary closed manifold M^{n+1}?

The answers to both questions rely on a scheme to find saddle point solutions of some variational problems – the min-max method.

General existence questions

Question 1:

Can we find a closed minimal hypersurface Σ^n in an arbitrary closed manifold M^{n+1} (without any assumption on homology, homotopy, or isotopy)?

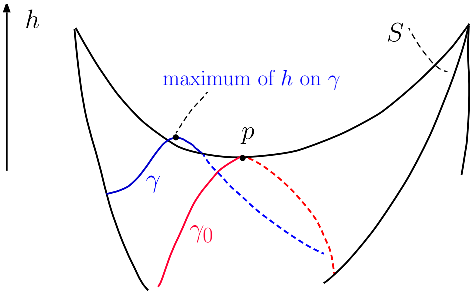
Question 2:

Can we find a closed CMC hypersurface Σ^n with an arbitrary prescribed curvature $c \in \mathbb{R}$ in an arbitrary closed manifold M^{n+1}?

The answers to both questions rely on a scheme to find saddle point solutions of some variational problems – the min-max method.

The idea of min-max method

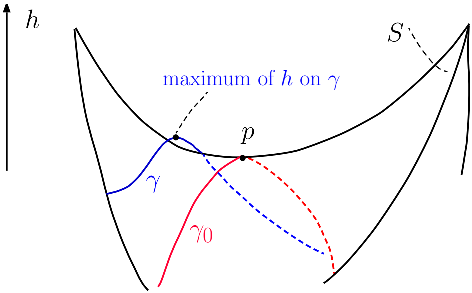
Multivariable Calculus: h is the height function on S, and p is a saddle point.



$$h(p) = \max_{t\in[0,1]} h\big(\gamma_0(t)\big) = \min_{\gamma\in[\gamma_0]} \max_{t\in[0,1]} h\big(\gamma(t)\big).$$

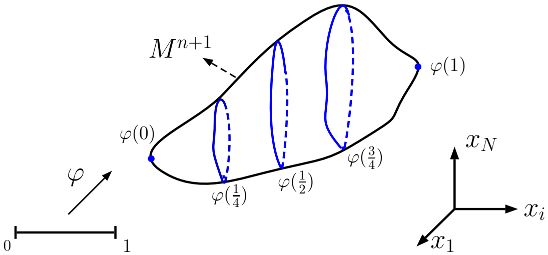
The idea of min-max method

Multivariable Calculus: h is the height function on S, and p is a saddle point.



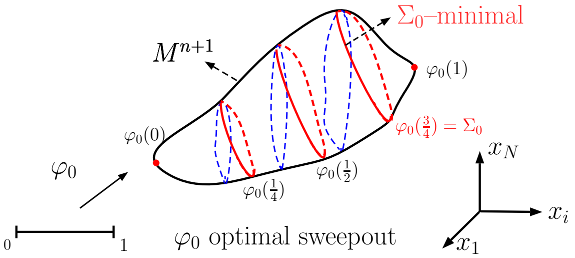
$$h(p) = \max_{t\in[0,1]} h\big(\gamma_0(t)\big) = \min_{\gamma\in[\gamma_0]} \max_{t\in[0,1]} h\big(\gamma(t)\big).$$

Almgren-Pitts and Schoen-Simon theory



- $\varphi : [0,1] \to$ space of hypercycles, – "sweepout";
- Min-max value —"width":

$$W = \inf\Big\{ \max_{t\in[0,1]} \text{Area}(\phi(t)) : \phi \text{ is a sweepout}\Big\}.$$

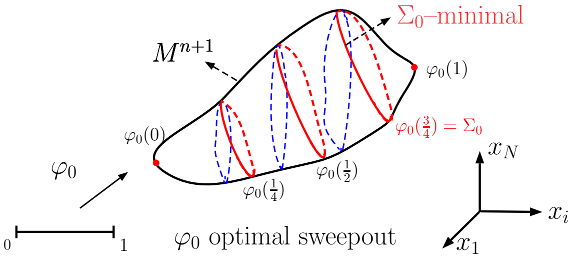


Theorem (Almgren 1965, Pitts 1981, Schoen-Simon 1981)

The width W is achieved as the area of some closed minimal hypersurface Σ_0 which is smoothly embedded outside a codim-7 singular set.

Therefore, every closed manifold admits a closed minimal hypersurface.

— This affirmatively answered Question 1.



Theorem (Almgren 1965, Pitts 1981, Schoen-Simon 1981)

The width W is achieved as the area of some closed minimal hypersurface Σ_0 which is smoothly embedded outside a codim-7 singular set.

Therefore, every closed manifold admits a closed minimal hypersurface.

— This affirmatively answered Question 1.

Topology of minimal surfaces

There were many works trying to control the topology of minimal surfaces.

- Simon-Smith 82: every Riemannian 3-sphere (S^3, g) admits an embedded minimal two sphere.
- Colding-De Lellis 03, DeLellis-Pellandini 10, Ketover 13: every closed 3-manifold M admits a closed embedded minimal surface with genus less than the Heegaard genus of M.
- Sacks-Uhlenbeck 81: if $\pi_k(M^n) \neq 0$, then (M, g) admits a branched immersed minimal two sphere.

Min-max theory for CMC hypersurfaces

Recall: "Question 2: Can we find a closed CMC hypersurface Σ^n with an arbitrary prescribed curvature $c \in \mathbb{R}$ in any closed M^{n+1}?"

Theorem (Z.- Zhu 17)

Given any closed (M^{n+1}, g) with $3 \leq (n+1) \leq 7$, then for any $c \in \mathbb{R}$, there exists a smooth, closed hypersurface Σ^n of constant mean curvature c.

- This generalizes the Almgren-Pitts-Schoen-Simon min-max theory for minimal hypersurfaces to the CMC setting ($c \neq 0$).
- This is the first general existence theory for closed CMC hypersurface with prescribed $c \in \mathbb{R}$.
- No topological control of Σ even when $n+1=3$ due to the use of geometric measure theory.

Min-max theory for CMC hypersurfaces

Recall: “Question 2: Can we find a closed CMC hypersurface Σ^n with an arbitrary prescribed curvature $c \in \mathbb{R}$ in any closed M^{n+1}?”

Theorem (Z.- Zhu 17)

Given any closed (M^{n+1}, g) with $3 \leq (n+1) \leq 7$, then for any $c \in \mathbb{R}$, there exists a smooth, closed hypersurface Σ^n of constant mean curvature c.

- This generalizes the Almgren-Pitts-Schoen-Simon min-max theory for minimal hypersurfaces to the CMC setting ($c \neq 0$).
- This is the first general existence theory for closed CMC hypersurface with prescribed $c \in \mathbb{R}$.
- No topological control of Σ even when $n + 1 = 3$ due to the use of geometric measure theory.

Min-max theory for CMC hypersurfaces

Recall: "Question 2: Can we find a closed CMC hypersurface Σ^n with an arbitrary prescribed curvature $c \in \mathbb{R}$ in any closed M^{n+1}?"

Theorem (Z.- Zhu 17)

Given any closed (M^{n+1}, g) with $3 \leq (n+1) \leq 7$, then for any $c \in \mathbb{R}$, there exists a smooth, closed hypersurface Σ^n of constant mean curvature c.

- This generalizes the Almgren-Pitts-Schoen-Simon min-max theory for minimal hypersurfaces to the CMC setting ($c \neq 0$).
- This is the first general existence theory for closed CMC hypersurface with prescribed $c \in \mathbb{R}$.
- No topological control of Σ even when $n+1=3$ due to the use of geometric measure theory.

Min-max theory for CMC hypersurfaces

Recall: "Question 2: Can we find a closed CMC hypersurface Σ^n with an arbitrary prescribed curvature $c \in \mathbb{R}$ in any closed M^{n+1}?"

Theorem (Z.- Zhu 17)

Given any closed (M^{n+1}, g) with $3 \leq (n+1) \leq 7$, then for any $c \in \mathbb{R}$, there exists a smooth, closed hypersurface Σ^n of constant mean curvature c.

- This generalizes the Almgren-Pitts-Schoen-Simon min-max theory for minimal hypersurfaces to the CMC setting ($c \neq 0$).
- This is the first general existence theory for closed CMC hypersurface with prescribed $c \in \mathbb{R}$.
- No topological control of Σ even when $n + 1 = 3$ due to the use of geometric measure theory.

We established a saddle point theory for a weighted area functional:

- If Σ bounds a domain $\Omega \subset M$, then Σ is CMC with mean curvature c iff it is a critical of the weighted area:

$$\mathcal{A}^c(\Omega) = \mathrm{Area}(\partial\Omega) - c\,\mathrm{Vol}(\Omega).$$

- In fact,

$$\frac{d}{dt}\Big|_{t=0} \mathcal{A}^c(\phi_t(\Omega)) = \int_{\partial\Omega} (\boldsymbol{H} - c)\boldsymbol{n} \cdot X.$$

  So $\delta\mathcal{A}^c|_\Omega = 0$ iff $\boldsymbol{H} \equiv c$.

- The c-min-max width is:

$$W^c = \inf\Big\{ \max_{t\in[0,1]} \mathcal{A}^c(\phi(t)) : \phi \text{ is a sweepout}\Big\}.$$

Min-max theory for PMC hypersurfaces

We later generalized our theory for prescribing varying mean curvature.

Theorem (Z.- Zhu 18)

Given any closed (M^{n+1}, g) *with* $3 \leq (n+1) \leq 7$, *then there exists a generic set of smooth functions* $\mathcal{S}$, *such that for any* $h \in \mathcal{S}$, *there exists a smooth, closed hypersurface* Σ^n *with prescribed mean curvature* h*; that is,*

$$\boldsymbol{H}_\Sigma = h|_\Sigma.$$

Min-max theory for CMC surfaces in 3-d

In dimension 3, we also considered the existence problem of CMC 2-spheres in 3-spheres, using a totally different PDE variational approach.

Thm A: (Cheng-Z. 20)

Given any (S^3, g), for almost all constant $c > 0$, there exists a branched immersed 2-sphere with constant mean curvature c.

Thm B: (Cheng-Z. 20)

If (S^3, g) has nonnegative Ricci curvature, there exists a branched immersed 2-sphere with mean curvature c for all constant $c > 0$.

Thm B partially solves a conjecture by Rosenberg-Smith in 2010.

Min-max theory for CMC surfaces in 3-d

In dimension 3, we also considered the existence problem of CMC 2-spheres in 3-spheres, using a totally different PDE variational approach.

Thm A: (Cheng-Z. 20)

Given any (S^3, g), for almost all constant $c > 0$, there exists a branched immersed 2-sphere with constant mean curvature c.

Thm B: (Cheng-Z. 20)

If (S^3, g) has nonnegative Ricci curvature, there exists a branched immersed 2-sphere with mean curvature c for all constant $c > 0$.

Thm B partially solves a conjecture by Rosenberg-Smith in 2010.

Min-max theory for CMC surfaces in 3-d

In dimension 3, we also considered the existence problem of CMC 2-spheres in 3-spheres, using a totally different PDE variational approach.

Thm A: (Cheng-Z. 20)

Given any (S^3, g), for almost all constant $c > 0$, there exists a branched immersed 2-sphere with constant mean curvature c.

Thm B: (Cheng-Z. 20)

If (S^3, g) has nonnegative Ricci curvature, there exists a branched immersed 2-sphere with mean curvature c for all constant $c > 0$.

Thm B partially solves a conjecture by Rosenberg-Smith in 2010.

Infinitely many solutions

Motivated by the Almgren-Pitts theory and results in dynamic systems about the existence of ∞-closed geodesics, Yau 1980s raised the following conjecture.

Yau's Conjecture on minimal surfaces

Every closed 3-manifold admits infinitely many distinct closed, immersed, minimal surfaces.

The solution of this conjecture used variational theory described in the following.

Infinitely many solutions

Motivated by the Almgren-Pitts theory and results in dynamic systems about the existence of ∞-closed geodesics, Yau 1980s raised the following conjecture.

Yau's Conjecture on minimal surfaces

Every closed 3-manifold admits infinitely many distinct closed, immersed, minimal surfaces.

The solution of this conjecture used variational theory described in the following.

Variational point of view

Variational approach

Find the right "space of hypersurfaces" with abundant topological structures, and search for critical points of the Area functional therein.

Example: critical points for quadratic forms

Let A be an $n \times n$ symmetric matrix. Its k-th eigenvalue is given by

$$\lambda_k = \min_{P \subset \mathbb{R}^n} \max_{x \in P, x \neq 0} Q_A(x), \text{ where } Q_A(x) = \frac{\langle Ax, x \rangle}{\langle x, x \rangle},$$

where P is a k-dimensional linear subspace.

Variational point of view

Variational approach

Find the right "space of hypersurfaces" with abundant topological structures, and search for critical points of the Area functional therein.

Example: critical points for quadratic forms

Let A be an $n \times n$ symmetric matrix. Its k-th eigenvalue is given by

$$\lambda_k = \min_{P \subset \mathbb{R}^n} \max_{x \in P, x \neq 0} Q_A(x), \text{ where } Q_A(x) = \frac{\langle Ax, x\rangle}{\langle x, x\rangle},$$

where P is a k-dimensional linear subspace.

Volume spectrum - I

Theorem (Almgren 1961)

The space of all closed separating hypersurfaces $\Sigma^n \subset M^{n+1}$, modulo the $\mathbb{Z}_2$-action on identifying the two orientations, satisfies:

$$\mathcal{Z}_n(M, \mathbb{Z}_2) = \{\Sigma = \partial\Omega \sim \partial(M \setminus \Omega)\} \simeq \mathbb{RP}^\infty.$$

Therefore, the $\mathbb{Z}_2$-cohomological ring is:

$$\mathcal{H}^*(\mathcal{Z}_n(M, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2[\lambda].$$

Definition

A k-sweepout ($k \in \mathbb{N}$) is a continue map: $\Phi : X \to \mathcal{Z}_n(M, \mathbb{Z}_2)$, such that

$$\Phi^*(\lambda^k) \neq 0 \in H^k(X, \mathbb{Z}_2).$$

Volume spectrum - I

Theorem (Almgren 1961)

The space of all closed separating hypersurfaces $\Sigma^n \subset M^{n+1}$, modulo the $\mathbb{Z}_2$-action on identifying the two orientations, satisfies:

$$\mathcal{Z}_n(M, \mathbb{Z}_2) = \{\Sigma = \partial\Omega \sim \partial(M \setminus \Omega)\} \simeq \mathbb{RP}^\infty.$$

Therefore, the $\mathbb{Z}_2$-cohomological ring is:

$$\mathcal{H}^*(\mathcal{Z}_n(M, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2[\lambda].$$

Definition

A *k-sweepout* ($k \in \mathbb{N}$) is a continue map: $\Phi : X \to \mathcal{Z}_n(M, \mathbb{Z}_2)$, such that

$$\Phi^*(\lambda^k) \neq 0 \in H^k(X, \mathbb{Z}_2).$$

Volume spectrum - I

Theorem (Almgren 1961)

The space of all closed separating hypersurfaces $\Sigma^n \subset M^{n+1}$, modulo the $\mathbb{Z}_2$-action on identifying the two orientations, satisfies:

$$\mathcal{Z}_n(M, \mathbb{Z}_2) = \{\Sigma = \partial\Omega \sim \partial(M \setminus \Omega)\} \simeq \mathbb{RP}^\infty.$$

Therefore, the $\mathbb{Z}_2$-cohomological ring is:

$$\mathcal{H}^*(\mathcal{Z}_n(M, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2[\lambda].$$

Definition

A k-sweepout ($k \in \mathbb{N}$) is a continue map: $\Phi : X \to \mathcal{Z}_n(M, \mathbb{Z}_2)$, such that

$$\Phi^*(\lambda^k) \neq 0 \in H^k(X, \mathbb{Z}_2).$$

Volume spectrum - II

Volume spectrum: Gromov 88, Guth 10, Marques-Neves 13

The k-th volume spectrum is

$$\omega_k(M,g) = \inf_{\Phi:k-\text{sweepout}} \max_{x\in X=\text{dom}(\Phi)} \text{Area}\big(\Phi(x)\big).$$

$$0 < \omega_1(M,g) \leq \cdots \omega_k(M,g) \sim k^{\frac{1}{n+1}} \to \infty.$$

Min-max Theorem

$\{\omega_k(M,g)\}$ are given by areas of closed minimal hypersurfaces (smoothly embedded when $2 \leq n \leq 6$) counted with multiplicity, i.e.

$$\omega_k = \sum_{i=1}^{l_k} m_i^k \,\text{Area}(\Sigma_i^k), \quad m_i^k \in \mathbb{N}_{>0}.$$

Volume spectrum - II

Volume spectrum: Gromov 88, Guth 10, Marques-Neves 13

The k-th volume spectrum is

$$\omega_k(M,g) = \inf_{\Phi: k-\text{sweepout}} \max_{x \in X = \text{dom}(\Phi)} \text{Area}\big(\Phi(x)\big).$$

$$0 < \omega_1(M,g) \leq \cdots \omega_k(M,g) \sim k^{\frac{1}{n+1}} \to \infty.$$

Min-max Theorem

$\{\omega_k(M,g)\}$ are given by areas of closed minimal hypersurfaces (smoothly embedded when $2 \leq n \leq 6$) counted with multiplicity, i.e.

$$\omega_k = \sum_{i=1}^{l_k} m_i^k \,\text{Area}(\Sigma_i^k), \quad m_i^k \in \mathbb{N}_{>0}.$$

Yau's conjecture and Multiplicity

The natural idea to solve Yau's conjecture was to apply the min-max theorem to the volume spectrum. However, the existence of possibly higher multiplicities may cause re-occurrence of minimal hypersurfaces, so one may not produce genuine new solutions!

- Marques-Neves 13: assume M^{n+1} $(2 \leq n \leq 6)$ has positive Ricci curvature, then every min-max solution is connected; existence of ∞-solutions follows from a counting argument.
- Song 18: when M admits disjoint closed minimal hypersurfaces, he cut the manifold into pieces and introduced a new notion of cylindrical volume spectrum; existence of ∞-solutions follows from a similar counting argument in a compact piece.

Yau's conjecture and Multiplicity

The natural idea to solve Yau's conjecture was to apply the min-max theorem to the volume spectrum. However, the existence of possibly higher multiplicities may cause re-occurrence of minimal hypersurfaces, so one may not produce genuine new solutions!

- Marques-Neves 13: assume M^{n+1} $(2 \leq n \leq 6)$ has positive Ricci curvature, then every min-max solution is connected; existence of ∞-solutions follows from a counting argument.
- Song 18: when M admits disjoint closed minimal hypersurfaces, he cut the manifold into pieces and introduced a new notion of cylindrical volume spectrum; existence of ∞-solutions follows from a similar counting argument in a compact piece.

Yau's conjecture and Multiplicity

The natural idea to solve Yau's conjecture was to apply the min-max theorem to the volume spectrum. However, the existence of possibly higher multiplicities may cause re-occurrence of minimal hypersurfaces, so one may not produce genuine new solutions!

- Marques-Neves 13: assume M^{n+1} $(2 \leq n \leq 6)$ has positive Ricci curvature, then every min-max solution is connected; existence of ∞-solutions follows from a counting argument.
- Song 18: when M admits disjoint closed minimal hypersurfaces, he cut the manifold into pieces and introduced a new notion of cylindrical volume spectrum; existence of ∞-solutions follows from a similar counting argument in a compact piece.

Multiplicity One Conjecture

Marques-Neves 14 raised the following conjecture, which is now a theorem.

Multiplicity One Conjecture (Z. 19)

Let M^{n+1} be a closed manifold with $2 \leq n \leq 6$. For a smooth generic metric g (in the sense of Baire), for each $k \in \mathbb{N}$, we have

$$\omega_k = \sum_{i=1}^{l_k} \text{Area}(\Sigma_i^k).$$

That is, all multiplicities are exactly 1.

- This is an analog of the basic fact in Morse theory: "generically a smooth function has only non-degenerate critical points".
- This directly implies Yau's conjecture under "generic assumption".

Multiplicity One Conjecture

Marques-Neves 14 raised the following conjecture, which is now a theorem.

Multiplicity One Conjecture (Z. 19)

Let M^{n+1} be a closed manifold with $2 \leq n \leq 6$. For a smooth generic metric g (in the sense of Baire), for each $k \in \mathbb{N}$, we have

$$\omega_k = \sum_{i=1}^{l_k} \text{Area}(\Sigma_i^k).$$

That is, all multiplicities are exactly 1.

- This is an analog of the basic fact in Morse theory: "generically a smooth function has only non-degenerate critical points".
- This directly implies Yau's conjecture under "generic assumption".

Morse Index Conjecture

By comparing with the classical Morse theory and the topological fact $\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2) \simeq \mathbb{RP}^\infty$, the area functional

$$\text{Area} : \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2) \to [0, \infty)$$

should contain as many critical points as the 'number of homologies'.

Morse Index Conjecture: Marque-Neves 14

In a generic (M^{n+1}, g) with $2 \leq n \leq 6$, for each $k \in \mathbb{N}$, there exists a closed minimal $\Sigma_k \subset M$ with

$$\text{Area}(\Sigma_k) = \omega_k(M, g), \text{ with Morse index } \text{Index}(\Sigma_k) = k.$$

Marques-Neves 16, 18 gave a strategy to solve this conjecture assuming the Multiplicity One Conjecture, so this is also a theorem now.

Higher multiplicity can appear!

Denote

$$S_a^{n+1} = \{x_1^2 + \cdots + x_{n+1}^2 + \frac{x_{n+2}^{2n}}{a^{2n}} = 0\} \subset \mathbb{R}^{n+2},$$

and $S_0^n = S_a^{n+1} \cap \{x_{n+2} = 0\}$.

Theorem (Wang-Z 22)

For $2 \leq n \leq 6$ and $a \gg 1$, the min-max minimal hypersurface associated with $\omega_2(S_a^{n+1})$ must be the equator S_0^n counted with multiplicity two.

Why could multiplicity arise?

Multiplicities could appear mainly as "convergence theory for minimal hypersurfaces" are involved.

A sequence of single-sheeted minimal hypersurfaces can collapse and result in a multi-sheeted limit.

Min-max with a Lagrange multiplier

- The way to resolve the issue was to use the perturbed functional:

$$\mathcal{A}^h(\Omega) = \text{Area}(\partial\Omega) - \int_\Omega h \, d\,\text{Vol}, \quad h : M \to \mathbb{R}.$$

  Min-max solutions w.r.t. $\mathcal{A}^h(\Omega)$ have multiplicity one.

- Changing $h \to \epsilon h$ and letting $\epsilon \to 0$, multiplicity one PMC hypersurfaces converge only to multiplicity one minimal hypersurfaces in a generic metric.

- To approximate Area by $\mathcal{A}^{\epsilon h}$, one needs to lift to the double cover $\mathcal{C}(M) \to \mathcal{Z}_n(M, \mathbb{Z}_2)$, and use a local min-max construction.

Multiplicity one for PMC functional

A critical point $\Sigma = \partial\Omega$ of $\mathcal{A}^h$ satisfies:

$$\boldsymbol{H}_\Sigma = h|_\Sigma.$$

Solutions of this equation satisfy the following one-sided “Maximum Principle”:

Some related open problems

Whether the Simon-Smith min-max theory satisfies a multiplicity one type conjecture?

Given a closed Riemannian manifold and a $H > 0$, prove the existence of two closed H-CMC hypersurfaces.

Extend the CMC min-max theory to suitable noncompact manifolds.

Thank you for your attention!