

Regular and singular minimal surfaces

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Plateau's problem

Problem

Given a 1-dimensional contour $\Gamma \subset \mathbb{R}^3$ find the surface Σ of least area which spans it.

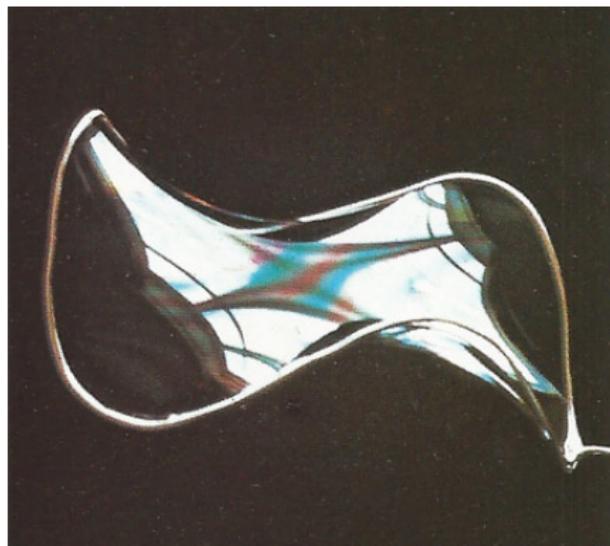
Named after the Belgian physicist Joseph Plateau



who proposed it to model the shapes of soap films in the XIX century.

Soap films

Take a wire, dip it in a bucket of soap water and pull it out.

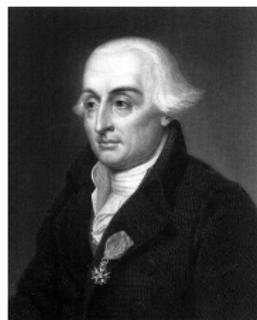


Calculus of variations

In mathematics “minimal surfaces” appear as early as 1760.

J. L. Lagrange, “Essai d’une **nouvelle méthode** pour déterminer les maxima et **les minima des formules integrales** indefinies.”

Miscellanea Taurinensia 2 325 (1760), 173–199

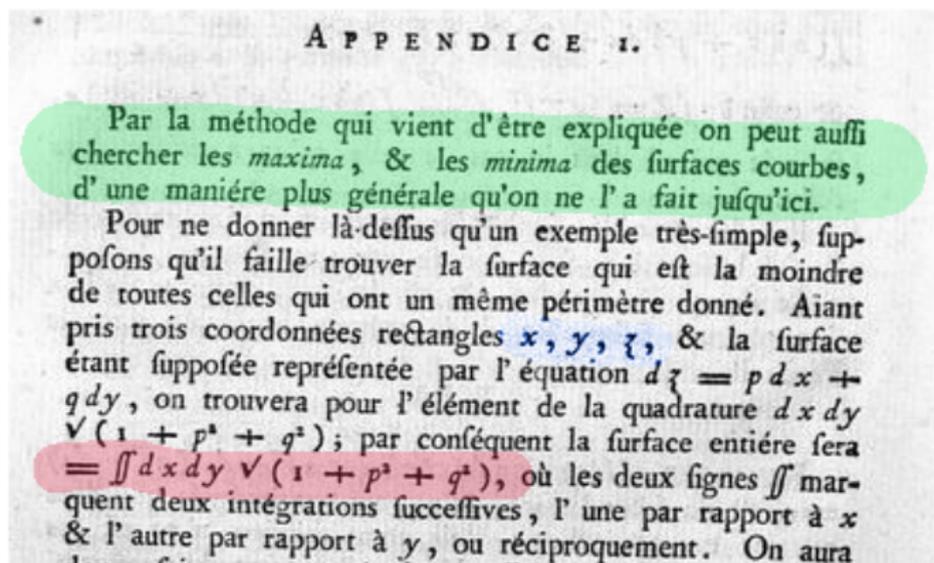


Lagrange shows how to compute the Euler-Lagrange equations for extrema of integral functionals.

Most of the work is dedicated to “1-dimensional problems”.

The area functional

But in the appendix he considers the area of 2-dimensional graphs



NB: $\zeta = \zeta(x, y)$,

$$p = \frac{\partial \zeta}{\partial x} \quad q = \frac{\partial \zeta}{\partial y}.$$

The minimal surface equation

He then derives the Euler-Lagrange equation for extrema

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 $\iint dx dy \left(\frac{dQ}{dy}\right) \delta z$. Substituant ces valeurs dans l'équation ci-dessus, elle deviendra $-\iint dx dy \left[\left(\frac{dP}{dx}\right) + \left(\frac{dQ}{dy}\right)\right] \delta z = 0$, laquelle devra être vraie indépendamment de δz ; on aura donc en général, pour tous les points de la surface cherchée, $\left(\frac{dP}{dx}\right) + \left(\frac{dQ}{dy}\right) = 0$; ce qui montre que cette quantité $P dy - Q dx$, favoir $\frac{P dy - q dx}{\sqrt{(1 + P^2 + q^2)}}$ doit être une différentielle complète. Le Problème se réduit donc à chercher p & q par ces conditions que $p dx + q dy$, & $\frac{p dy - p dx}{\sqrt{(1 + p^2 + q^2)}}$ soient l'une & l'autre des différentielles exactes.

$p dx + q dy$ and $\frac{-q dx + p dy}{\sqrt{1 + p^2 + q^2}}$ are exact forms

The minimal surface equation

Equivalently, in modern notation,

$$\operatorname{div} \frac{\nabla \zeta}{\sqrt{1 + |\nabla \zeta|^2}} = 0$$

(Lagrange argues locally, i.e. on a simply connected domain!)

This might very well be (historians to the rescue, please!):

- ▶ The first multi-dimensional Euler-Lagrange condition ever derived.
- ▶ The first elliptic partial differential equation ever written.

Back to the Plateau problem

Problem

Given an $(m - 1)$ -dimensional surface Γ in an $(m + n)$ -dimensional Riemannian ambient manifold \mathcal{M} , find the m -dimensional surface Σ which spans Γ and has *least volume*.

But what should we understand with the words **surface**, **spans**, and **volume**?

One classical answer (Douglas-Rado, thirties):

- ▶ fix an abstract manifold Σ' ;
- ▶ look at $\Phi : \Sigma' \rightarrow \mathcal{M}$ (sufficiently regular, e.g. cont. differentiable) s.t. $\Phi|_{\partial\Sigma'}$ is homeo onto Γ .

Geometric measure theory

GMT offers two “philosophical approaches” to Plateau’s problem.

Set theoretic (e.g. Reifenberg 1960):

- ▶ Σ is any (closed) set;
- ▶ volume is a measure (e.g. the Hausdorff measure);
- ▶ bounds means “ Γ is trivial in Σ ” in some topological sense (e.g. in Čech homology, Reifenberg).

Functional-analytic (e.g. De Giorgi, 1953-1958):

- ▶ fix a class of desired Σ , with known boundary and known volume (e.g. boundaries of smooth open sets, De Giorgi);
- ▶ decide a (good) topology;
- ▶ complete the space;
- ▶ extend boundary as a cont. operation, extend volume as a lower semicontinuous function.

Good means **direct methods of the calculus of variations apply.**

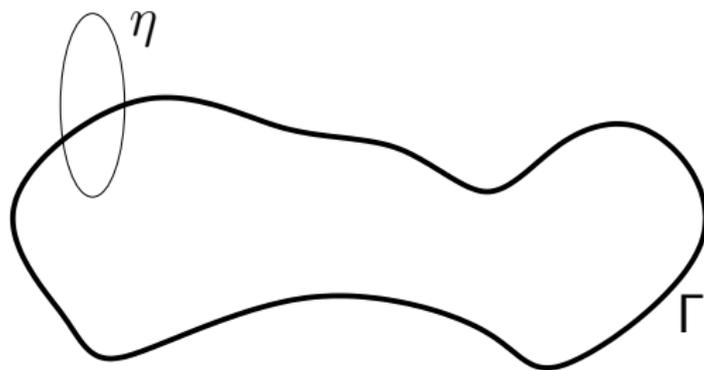
An example of set-theoretic approach

$\Gamma \subset \mathbb{R}^3$ closed curve

Σ closed set

Definition (Harrison 2011)

Σ spans Γ if it intersects every smooth closed curve η whose linking number with Γ is 1.



An example of functional-analytic approach

Start: **singular chains**, coefficients in a (normed) **group** G . E.g. $G = \mathbb{Z}$.

$$\Sigma = \sum_i k_i \Sigma_i$$

$\Sigma_i =$ **smooth image of an m -dimensional simplex**, $k_i \in G$ (e.g. \mathbb{Z}).

- ▶ ∂ is the usual boundary,
- ▶ Volume is the usual volume (careful if Σ_i overlap!).

Topology? Model case: $\Sigma^{(1)}, \Sigma^{(2)} \subset \mathbb{R}^{m+n}$ have **equal boundary**.

$$\Sigma^{(1)} - \Sigma^{(2)} = \partial \Lambda$$

$$d(\Sigma^{(1)}, \Sigma^{(2)}) = \inf \{ \text{Volume}(\Lambda) \}.$$

Oriented Plateau problem: Federer-Fleming theory

Federer-Fleming 1960: The completion is the space of “flat chains”.
For the Plateau’s problem: the subspace of **integral currents**.

An **integer rectifiable** current consists of a **series** $\sum_i k_i K_i$

- ▶ $K_i \subset \Gamma_i$ pairwise disjoint **compact subsets**
- ▶ $\Gamma_i \subset \mathbb{R}^{m+n}$ C^1 m -dim. oriented submanifolds
- ▶ k_i **positive integers**

with

$$\sum_i k_i \text{Vol}^m(K_i) < \infty \quad (1)$$

T is an **integral current** if both T and ∂T are integer rectifiable.

Federer-Fleming 1960: **Direct methods apply!**

Other **groups** G possible, e.g. $G = \mathbb{Z}/p\mathbb{Z}$.

Definition

A set E is m -rectifiable if it can be chopped into

- ▶ countably many closed pieces K_i contained in m -dimensional C^1 submanifolds Γ_i
- ▶ plus a “residual m -null” set K_0 (i.e. K_0 has zero m -dimensional Hausdorff measure).

m -rectifiable sets are “measure-theoretic substitutes” of C^1 m -dimensional submanifolds

From m -rectifiable set to integer rectifiable current add: orientation, integer multiplicities, and finite m -dimensional measure.

Comparisons and varifolds

Different approaches have different advantages. Typically:

- ▶ Set-theoretic better suited for modeling soap films (cf. Maggi-Scardicchio-Stuvard 2018, King-Maggi-Stuvard 2019)
- ▶ Functional analytic better suited to study geometry of manifolds.

(Integral) Varifolds, introduced by Almgren: (rectifiable) measures on the Grassmannian of unoriented m -dimensional planes.

Powerful notion of generalized surfaces, which can be **deformed**

- ▶ **Stationary** (vanishing first variation)
- ▶ **Stable** (stationary + nonnegative second variation)

Minimizers in all approaches give **stable varifolds**.

Singularities (in the oriented Plateau problem)

Theorem (Bombieri-De Giorgi-Giusti 1969)

Consider the cone over $\frac{\sqrt{2}}{2}\mathbb{S}^3 \times \frac{\sqrt{2}}{2}\mathbb{S}^3 \subset \mathbb{S}^7 = \partial\mathbf{B}_1 \subset \mathbb{R}^8$ (*Simons' cone*):

$$\Sigma := \{x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}.$$

Any *smooth oriented 7-dimensional surface* $\Sigma' \subset \mathbb{R}^8$ with $\partial\Sigma' = \Sigma \cap \partial\mathbf{B}_1$ *has larger volume* than $\Sigma \cap \mathbf{B}_1$.

Theorem (Federer 1965 (Wirtinger 1936))

Consider

$$\Sigma = \{z = (z_1, z_2) \in \mathbb{C}^2 : z_1^2 = z_2^{2Q+1}\} \subset \mathbb{R}^4.$$

Any *smooth oriented 2-(real!)-dimensional surface* $\Sigma' \subset \mathbb{R}^4$ with $\partial\Sigma' = \Sigma \cap \partial\mathbf{B}_1$ *has larger area* than $\Sigma \cap \mathbf{B}_1$.

- ▶ When can we rule out singularities?
- ▶ How large can the singular set be (at most)?
- ▶ Which structure does it have?

	Station.	Stable	Min. type 1	Min. type 2	...
Int.					
Bdry					

Plan for the rest of the talk

- ▶ Allard's theory for stationary varifolds;
- ▶ Set-theoretic solutions: Taylor's theorem and more;
- ▶ Hypercurrents
 - ▶ \mathbb{Z} : Interior regularity
 - ▶ \mathbb{Z} : Boundary regularity
 - ▶ Stability
 - ▶ $\mathbb{Z}/p\mathbb{Z}$: Interior regularity
- ▶ Currents in codimension > 1 , interior
 - ▶ \mathbb{Z} : Almgren's interior regularity theory
 - ▶ $\mathbb{Z}/p\mathbb{Z}$.
- ▶ Integral currents in codimension > 1 , boundary
 - ▶ Regularity: state of the art
 - ▶ Challenges and conjectures
- ▶ Rectifiability of the interior singular set in codimension > 1
- ▶ Uniqueness of tangent cones and beyond rectifiability

Allard's ε -regularity theorems

Most general, stationary (integral) varifolds.

Theorem (Allard's interior regularity, 1972)

V stationary varifold, q interior point.

If in $\mathbf{B}_{2r}(q)$ V is *close enough* to a *disk* (with multiplicity 1) then it is a *smooth perturbation of it* in $\mathbf{B}_r(q)$.

Useful because *close* is meant in a very *weak sense*.

Predated by ε -regularity theorems of De Giorgi 1961 (oriented hypersurfaces) and Reifenberg 1964 (his set-theoretic solutions)

Theorem (Allard's boundary regularity, 1975)

V stationary varifold, q boundary point, boundary Γ smooth

If in $\mathbf{B}_{2r}(q)$ V is *close enough* to a *half disk* (with multiplicity 1) then it is a *smooth perturbation of it* in $\mathbf{B}_r(q)$.

Corollaries of Allard's theory

Corollary (Interior theory)

The interior singular set is *meager* (i.e. the interior regular set is open and dense).

Open problem (Since 50 years!)

Show that the interior singular set has smaller size.

Corollary (Boundary theory)

U smooth bounded uniformly convex open set

smooth $\Gamma \subset \partial U$ (*convex barrier*)

T area minimizing integral current spanning Γ (solution of the *oriented Plateau problem*).

Then T is *regular in a neighborhood of Γ* .

What's the idea?

Key starting point of pretty much **all** ε -regularity theories (till now), first appearing in De Giorgi 1961.

The **linearization of the min. surface equation** is the **Laplace equation!**
E.g. for graphs of $u : \Omega \rightarrow \mathbb{R}$,

$$\int \sqrt{1 + |\nabla u|^2} = \int \left(1 + \frac{1}{2} |\nabla u|^2 + O(|\nabla u|^4) \right).$$

More involved for $u : \Omega \rightarrow \mathbb{R}^n$:

$$\int \sqrt{1 + |\nabla u|^2 + \sum_{M \subset Du, k \times k, \text{ minor}, k \geq 2} (\det M)^2}$$

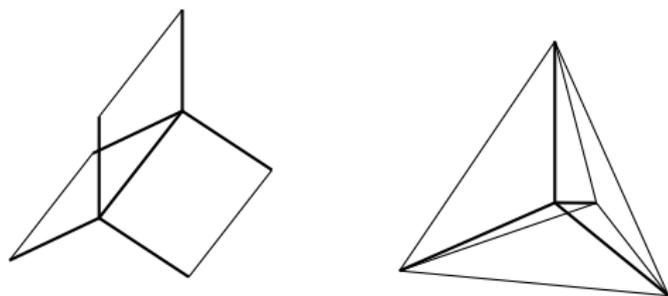
Set-theoretic minimizers, Taylor's theorem

The beautiful structure theorem of Jean Taylor

Theorem (Taylor 1976)

$\Sigma \subset \mathbb{R}^3$ set-theoretic 2d-minimizer, q interior point

(At a sufficiently small scale) at q , Σ is diffeomorphic to either a disk, or a triple junction, or a tetrahedral cone.



White 1985, Simon 1993, Colombo-Edelen-Spolaor 2017, results in more general dimension/codimension.

Set-theoretic minimizers, boundary regularity

Σ 2d set-theoretic minimizer, q boundary point, Γ smooth curve

Allard's Theorem:

- ▶ closeness to half disk \implies diffeo to model

David 2014:

- ▶ closeness to two half disks \implies diffeo to model
- ▶ closeness to triple junction \implies diffeo to model

Open problem

Complete boundary regularity theory.

Minimizing oriented hypersurfaces, interior regularity

Theorem (De Giorgi 1961, Fleming, Almgren, Simons, Federer 1970)

T area-minimizing m -dimensional integral current in \mathbb{R}^{m+1} (or in a “suitable” Riemannian manifold).

The set of interior singularities $\text{Sing}_i(T)$ has (Hausdorff) $\dim. \leq m - 7$.

Note: optimal! Cf. Bombieri-De Giorgi-Giusti 1969 (Simons' cone).

More structure?

Theorem

- ▶ Federer 1970: $m = 7$, isolated singularities
Simon 1983, “unique model” (more later);
- ▶ Simon 1993: $m > 7$, singular set rectifiable (“almost C^1 ”)
Naber-Valtorta 2015, finite $(m - 7)$ -dimensional volume.

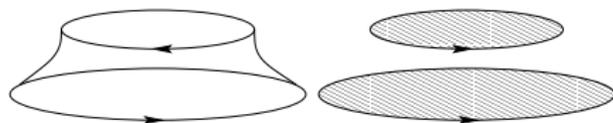
Minimizing oriented hypersurf., boundary regularity

Completely resolved by

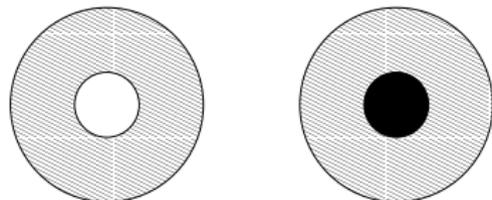
Theorem (Hardt-Simon 1980)

An m -dimensional area-minimizing current T in \mathbb{R}^{m+1} with a smooth multiplicity 1 boundary Γ is smooth in a neighborhood of Γ .

Need to understand “smoothness”...



Move the circles towards each other until they become coplanar.



Stable hypersurfaces

Schoen-Simon-Yau 1975: a-priori estimates for stable hypersurfaces

Theorem (Schoen-Simon 1981)

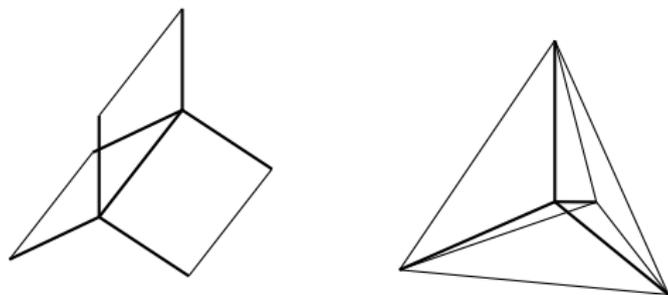
V stable m -dim. varifold in \mathbb{R}^{m+1} (or Riem. manifold).

$\text{Sing}(V)$ is $(m-2)$ -dim. null set $\implies \text{Sing}(V)$ has dimension $\leq m-7$.

Wickramasekera: deep theory for the optimal assumption.

Theorem (Wickramasekera 2014)

$\text{Sing}_i(V)$ is $(m-1)$ -dim. null set $\implies \text{Sing}_i(V)$ has dimension $\leq m-7$.



Open problem

A complete regularity theory for stable hypervarifolds without additional assumptions on $\text{Sing}_i(V)$ (e.g. $\text{Sing}_i(V)$ has dimension at most $m - 1$).

Open problem

Boundary regularity theory.

White's blow-up trick: a-priori estimates if the boundary has multiplicity 1 and there is a convex barrier, cf. D-Pellandini 2010.

Minimizing hypersurfaces mod p

Coefficient group $\mathbb{Z}/p\mathbb{Z}$.

Theorem

T area minimizing m -dimensional hypercurrent mod p .

$p = 2 \implies$ the **dimension** of $\text{Sing}_i(T)$ is $\leq m - 7$.

$p > 2 \implies \text{Sing}_i(T)$ is a C^1 $m - 1$ -dimensional submanifold
+ $(m - 2)$ -dim. (“more singular”) exceptional set.

$p = 3$ Taylor 1973 (and residual exception $\leq m - 3$ dimension)

p odd D-Hirsch-Marchese-Stuvard-Spoloar 2021

(building on White 1985 and Simon 1993)

Minter-Wickramasekera 2021 (from W 's stable varifold theory)

$p = 4$ White 1979

p even DHMSS 2022 (building on MW 2021, DHM-Stuvard 2019

and Almgren 1980, D-Spadaro 2009-2016, **more later**)

Oriented minimizing surfaces, Almgren's theory

In 1980 Almgren types his “Big Regularity paper”, 1728 pages

Theorem

T area-minimizing m -dim. integral current in \mathbb{R}^{m+n} (or Riem. manifold).
Then $\text{Sing}_j(T)$ has **dimension $\leq m - 2$** .

The theorem is optimal, the dimension of $\text{Sing}_j(T)$ matches Federer's examples (holomorphic subvarieties!)

What makes Almgren's theorem so difficult?

Almgren's theory: no ε -regularity

Consider $\Sigma = \{(z, w) \in \mathbb{C}^2 : z^2 = w^3\}$ (Federer \implies minimizing in \mathbb{R}^4).

“Zoom” around 0: $\mathbf{B}_r(0) \ni x \mapsto \frac{x}{r} \in \mathbf{B}_1(0)$.

$$\Sigma \mapsto \Sigma_r = \{(z, w) : z^2 = rw^3\}$$

Σ_r is closer and closer to a double copy of $\{z = 0\}$ as $r \downarrow 0$

But never diffeomorphic to it! 0 is a **singular flat point!**

Much worse: $\{(z - w^Q)^2 = w^{2Q+1}\}$

Almgren's theory in one slide

T area-minimizing integral current, 0 singular flat point

Then at any scale around 0 where T is sufficiently flat there are:

- ▶ an “osculating smooth manifold” \mathcal{M} to T (the center manifold)
e.g. the $z = w^Q$ part in $\{(z - w^Q)^2 = w^{2Q+1}\}$
- ▶ and a Q -sheeted “harmonic” approximation of T
over the normal bundle of \mathcal{M} .

The harmonic approximation is nontrivial, which obstructs the clustering of too many singular points around 0.

D-Spadaro 2009-2016: A more efficient treatment of Almgren's theory.

- ▶ Modern technology to treat multivalued harmonic approximation
- ▶ Shorter/more flexible construction of the center manifold \mathcal{M} .

Beyond Almgren's theory

Theorem (Chang 1988)

T area-minimizing *2-dim.* integral current in \mathbb{R}^{2+n} (or Riem. manifold).
Then $\text{Sing}_j(T)$ consists of *isolated points*.

In fact Chang needs a suitable “*upgrade*” of the center manifold. Its existence has been proved in D-Spadaro-Spolaor 2018.

Open problem

T area-minimizing m -dim. integral current in \mathbb{R}^{2+n} (or Riem. manifold).
Is $\text{Sing}_j(T)$ $(m - 2)$ -rectifiable?

More on this later.

Minimizing surfaces mod p

Theorem

T area-minimizing m -dim. current mod p in \mathbb{R}^{m+n} (or Riem. manifold)

$p = 2$ Federer 1970, $\text{Sing}_i(T)$ has **dimension** $\leq m - 2$

$p > 2$ D-Hirsch-Marchese-Stuvard 2019,
 $\text{Sing}_i(T)$ has **dimension** $\leq m - 1$

$p = 2$ can be achieved with ε -regularity theory.

$p > 2$ needs an “Almgren’s type theory”

Further, for $p = 2$ Simon’s techniques apply, for p **odd** Naber-Valtorta’s techniques apply: $\text{Sing}_i(T)$ is rectifiable.

Open problem

Prove the $(m - 1)$ -rectifiability for $p > 2$ even
(proved only in codimension $n = 1$).

White 1979: for **even** $p > 2$ area-minimizing **hyper**currents mod p can have **flat singular points**.

This case needs an “Almgren-type theorem” even in codimension 1 (D-Hirsch-Marchese-Stuvard-Spoloar 2022).

Higher codimension, boundary regularity

Singularities can occur even for the oriented Plateau problem

For instance, in $\mathbb{R}^4 = \mathbb{C}^2$, consider

$$\Sigma = \underbrace{\{z_2 = 0, \operatorname{Re} z_1 > 0\}}_{\text{a half plane}} \cup \underbrace{\{z_1 = 0\}}_{\text{a plane}} .$$

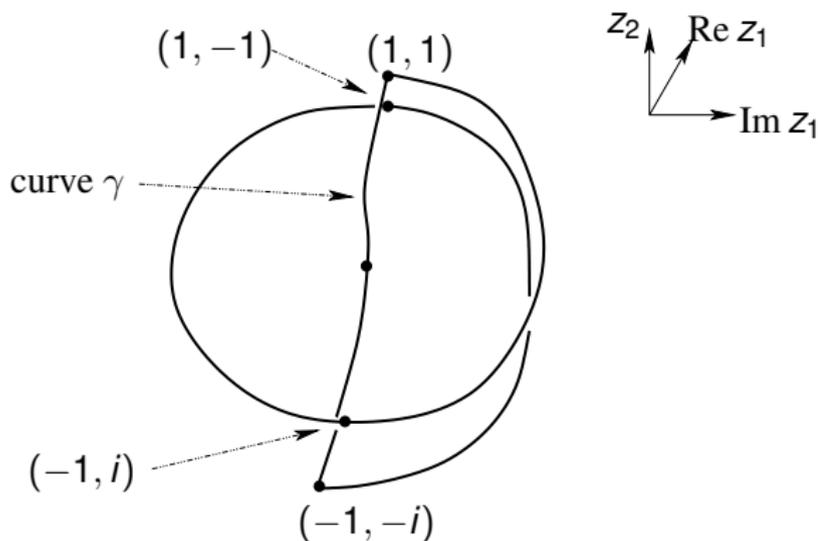
$\Sigma \cap \mathbf{B}_1$ is area minimizing
the boundary is:

- ▶ a segment through the origin “closed” by a half great circle
- ▶ plus a (disjoint) great circle

and the origin is a boundary singular point

We will “carve” a nastier example in $\{z \in \mathbb{C}^2 : z_1^2 = z_2^Q\}$, $Q = 2k + 1$.

A boundary branch point



$$\gamma(t) = \begin{cases} (-t, -i|t|^{(2k+1)/2}) & \text{for } t < 0 \\ (t, t^{(2k+1)/2}) & \text{for } t \geq 0. \end{cases}$$

Recall Allard's boundary regularity theorem
(1969 codim. 1, 1975 any codimension)

Theorem

If Γ lies in the boundary of a smooth uniformly convex set Ω , then T has no boundary singularities.

Elementary: T is contained in the convex hull of Γ .

\implies at a boundary point q T must be contained on one side of the hyperplane tangent to $\partial\Omega$

\implies boundary points are one-sided \implies regularity

Multiplicity 1, any codimension, general Γ

What happens without a convex barrier? Needs an “Almgren-type theory” at the boundary.

Theorem (D, De Philippis, Hirsch, Massaccesi 2018)

T m -dim. area-minimizing integral current in \mathbb{R}^{m+n} (or Riem. manifold), $\partial T = \llbracket \Gamma \rrbracket$: then the *set of boundary reg. points is open and dense* in Γ .

Answers a question of Almgren 1980, but it seems sort of lame ...
in fact *close to optimal!*

Example (DDHM 2018)

There is a 1-dimensional Γ in \mathbb{R}^4 and an area-minimizing current T with $\partial T = \llbracket \Gamma \rrbracket$ whose boundary singularities form a (Cantor-like) 1-dimensional set!

These singularities are “mild” (T is locally *decomposable in intersecting smooth pieces*).

Some conjectures

Conjecture (DDHM 2018)

For multiplicity 1 boundaries boundary singular points have zero $(m - 1)$ -dimensional measure.

“Essential (e.g. branching)” singularities have dimension $m - 2$.

Conjecture (White 1990)

T area minimizing 2-dimensional current in \mathbb{R}^{2+n} with $\partial T = \llbracket \Gamma \rrbracket$ for some closed **real analytic curve** Γ . Each boundary singularity is isolated.

Theorem (White)

If the conjecture holds T is a classical branched surface with finite topology and all the branch points are in the interior.

A worse example

Theorem (D, De Philippis, Hirsch 2019)

There are:

- ▶ a C^∞ closed Riemannian 4-manifold \mathcal{M}
- ▶ a C^∞ closed simple curve $\Gamma \subset \mathcal{M}$
- ▶ an area minimizing 2-dimensional current T in \mathcal{M} bounding Γ

with an infinite number of boundary branch points.

Corollary (DDH)

There are \mathcal{M} and $\Gamma \subset \mathcal{M}$ bounding a unique area-minimizing oriented surface T with *infinite topology*.

For the latter Γ the Douglas-Rado approach will keep producing more and more complicated surfaces as we increase the genus of the model, never reaching the absolute minimum.

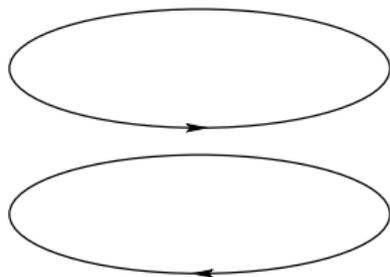
Higher multiplicity at the boundary

Open problem (Allard 1969, White 1983)

Develop a regularity theory for oriented minimizers at smooth boundaries with multiplicity higher than 1.

NB, it is false that the minimizer has necessarily the same multiplicity of the boundary!

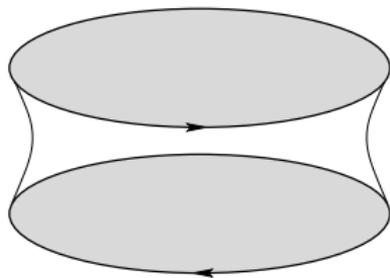
Consider $\Gamma = \Gamma_1 \cup \Gamma_2$ the union of two planar parallel circles in \mathbb{R}^3 , with opposite orientation.



If suitably arranged, Γ bounds two distinct area minimizing integral currents T_1 and T_2 .

An example

- ▶ T_1 consists of two planar disks;
- ▶ T_2 is a catenoid.



$T = T_1 + T_2$ is an area minimizing current with $\partial T = 2[\Gamma]$.

At any boundary point we see (locally) two smooth surfaces “departing” transversally from the boundary.

In codimension 1: higher multiplicities can be reduced to multiplicity 1 case thanks to a general decomposition theorem.

The decomposition theorem is false in higher codimension (L.C. Young 1963)

2-dimensional minimizers in higher codimension:

D-Nardulli-Steinbruechel 2021:

Allard's type theorem, full regularity with convex barrier

Nardulli-Resende De Oliveira 2022:

boundary regular set is dense in general

Back to interior regularity

Recall:

Theorem (Almgren)

T area-minimizing m -dim. integral current in \mathbb{R}^{m+n} (or Riem. manifold).
Then $\text{Sing}_j(T)$ has dimension $\leq m - 2$.

Open problem

Is $\text{Sing}_j(T)$ $(m - 2)$ -rectifiable?

Recall: having fixed a flat singular point q , at any scale at which T is sufficiently flat there are:

- ▶ an “osculating smooth manifold” \mathcal{M} to T (the center manifold)
- ▶ and a Q -sheeted “harmonic” approximation of T over the normal bundle of \mathcal{M} .

Theorem (D-Skorobogatova 2022)

The Q -sheeted approximation can be taken radially homogeneous.

The homogeneity is ≥ 1 and depends on q but not on the scale: there is a unique singularity degree of T at q .

If degree > 1 , T is flat at all scales and there is a unique approx. plane

If degree $> 2 - \delta$, the center manifold is the same at all scales

Forthcoming: flat singular points with singularity degree > 1 form an $(m - 2)$ -rectifiable set
(based on D-Marchese-Spadaro-Valtorta 2016, Naber-Valtorta 2014).

From rectifiability to uniqueness of tangent cone

Big challenge: when **degree = 1** the approx. plane **might depend on the scale**.

Monotonicity formula+compactness: at any point q , homothetic rescalings of any **generalized minimal surface** converge, **up to subsequences (!)**, to a **generalized minimal cone**.

Open problem (Uniqueness of tangent cones)

Is the limit independent of the subsequence?

Important cases solved by Taylor 1976, Allard-Almgren 1981 and in a fundamental work of Simon 1983 (see also further works 1993, 1995).

Theorem (Simon 1983)

The cone is unique at isolated singularities if it has multiplicity 1.

Widely open for **cones with multiplicity > 1** , except for White 1982: **2-dim.** area-minimizing integral currents.

Beyond rectifiability

Can we hope for better than rectifiability? **Yes** in some cases, see **Taylor's structure theorem**, or **hypersurfaces mod p** . **But not in general!**

Theorem (Simon 2021)

For any $\alpha \leq m - 7$ there are

- ▶ a **smooth Riemannian manifold \mathcal{M}** of dimension $m + 1$;
- ▶ and a stable minimal **hypersurf. $\Sigma \subset \mathcal{M}$** with $\dim(\text{Sing}_j(\Sigma)) = \alpha$.

Theorem (Liu 2022)

For any $\alpha \leq m - 2$ there are

- ▶ a **smooth Riemannian manifold \mathcal{M}** of dimension $m + n$;
- ▶ and an **m -dimensional area-minimizing integral current T in \mathcal{M}** with $\dim(\text{Sing}_j(T)) = \alpha$.

NB: \mathcal{M} is C^∞ but not real analytic.

**Thank you
for your attention!**