The number of closed ideals in the algebra of bounded operators on Lebesgue spaces

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Based on two papers, one with Bill Johnson and Gilles Pisier

and one with Bill Johnson

Ideals in $L(X)$

$L(X)$ is the Banach algebra of bounded linear operators on the Banach space X.

A closed ideal in $L(X)$ is a closed subspace $\mathcal{I}$ of $L(X)$ such that for all $T \in L(X)$ and $S \in \mathcal{I}$, TS and ST are in $\mathcal{I}$.

For $1 \le p < \infty$,

$$L_p = L_p[0,1] = \{f : [0,1] \to \mathbb{R}; \|f\|_p = \Big(\int_0^1 |f|^p\Big)^{1/p} < \infty\},$$

$$\ell_p = \{\{a_n\}_{n=1}^{\infty}; \Big(\sum_{n=1}^{\infty} |a_n|^p\Big)^{1/p} < \infty\}$$

Ideals in $L(X)$

There are some classical closed ideals in $L(X)$. $\overline{F(X)}$, the closure of the finite rank operators is contained in any closed ideal. $K(X)$ the set of compact operators (is equal to $\overline{F(X)}$ in classical spaces but not always). Another is $W(X)$, the set of weakly compact operators; operators T that map the unit ball into a weakly compact set. So $W(X) = L(X)$ iff X is reflexive. Another closed ideal is $S(X)$, the space of strictly singular operators on X. An operator T is strictly singular if it is not an into isomorphism when restricted to any infinite dimensional subspace.

Ideals in $L(X)$

A maximal algebraic ideal is automatically closed since the invertible elements in a Banach algebra form an open set, so every (always proper) closed ideal is contained in a closed maximal ideal. What are the maximal ones? Is there even a largest ideal?

It is known, but non trivial, that in $L(L_p)$ there is a unique maximal ideal $\mathcal{M}(L_p)$ and that it is equal to the set of L_p-singular operators, that is the set of operators that are not an isomorphism when restricted to any subspace isomorphic to L_p.

ideals in $L(X)$

A common way of constructing a (not necessarily closed) ideal in $L(X)$ is to take some operator $U : Y \to Z$ between Banach spaces and let $\mathcal{I}_U$ be the collection of all operators on X that factor through U, i.e., all $T \in L(X)$ s.t. there exist $A \in L(X, Y)$ and $B \in L(Z, X)$ s.t. $T = BUA$.

$L(X)\mathcal{I}_U L(X) \subset \mathcal{I}_U$ is clear, so $\mathcal{I}_U$ is an ideal in $L(X)$ if $\mathcal{I}_U$ is closed under addition. One usually guarantees this by using a U s.t. $U \oplus U : Y \oplus Y \to Z \oplus Z$ factors through U

Another way is, giving a set of operators $\{T_\alpha\} \subset L(X)$, look at the set of all finite sums of the form $\sum_{i=1}^n A_i T_{\alpha_i} B_i,\ A_i, B_i \in L(X)$.

Large and Small Ideals

$S(X)$: Strictly singular operators on X.

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

So, for example, $\overline{\mathcal{I}}_U$ is small if U is strictly singular and $U \oplus U$ factors through U.

And, for example, $\overline{\mathcal{I}}_U$ is large if $U = I_Y$ for some complemented subspace (= range of an idempotent) Y of X and $Y \oplus Y$ is isomorphic to Y.

To simplify notation, I'll write $\mathcal{I}_Y$ instead of $\mathcal{I}_{I_Y}$.

Ideals in $L(L_1)$

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

Small closed ideals in $L(L_1)$ include $K(L_1)$, $S(L_1)$, and $W(L_1)$. But $W(L_1) = S(L_1)$ Dunford-Pettis property of L_1.

Large closed ideals in $L(L_1)$ include $\overline{\mathcal{I}}_{\ell_1}$ and the largest ideal $\mathcal{M}(L_1)$ (and also the Dunford–Pettis opertors).

Incidently, Every large ideal in $L(L_1)$ contains $\overline{\mathcal{I}}_{\ell_1}$ and $\overline{\mathcal{I}}_{\ell_1}$ contains any small ideal in $L(L_1)$.

Until recently this is all that was known. This led Pietsch to ask in his 1979 book "Operator Ideals" whether there are infinitely many closed ideals in $L(L_1)$.

Ideals in $L(L_1)$ - the difficulty

It is easy to build closed ideals in $L(X)$; in particular, in $L(L_1)$; but difficult to prove that ideals are different. For example, for $1 < p < \infty$, let $\mathcal{I}_{L_p}$ be the (non closed) ideal of operators on L_1 that factor through L_p. These are all different, but their closures $\overline{\mathcal{I}}_{L_p}$ are all equal to the weakly compact operators on L_1.

One would guess that the key to solving Pietsch's problem was to find just one new closed ideal in $L(L_1)$. A few years ago Bill Johnson and I did that. The ideal is the closure of $\mathcal{I}_{J_2}$, where $J_2 : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto the Rademacher functions IID Bernoulli random variables that take on the values 1 and -1, each with probability $1/2$. We were excited when we were able to prove that $\overline{\mathcal{I}}_{J_2}$ is different from the previously known ideals. We then looked at $\overline{\mathcal{I}}_{J_p}$, $1 < p < 2$, where $J_p : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto IID p-stable random variables. The ideals $\mathcal{I}_{J_p}$ are all different, but it turns out that all the $\overline{\mathcal{I}}_{J_p}$ are equal to $\overline{\mathcal{I}}_{J_2}$!

Ideals in $L(L_1)$

Theorem. (Johnson, Pisier, Schechtman 2020)

There are at least $2^{\aleph_0}$ (small) closed ideals in $L(L_1)$. Moreover, these ideals are even mutually non-isomorphic as Banach algebras. The same is true in $L(L_\infty)$.

It remains open whether there are infinitely many large closed ideals in $L(L_1)$. This is connected to the unsolved problem whether every infinite dimensional complemented subspace of L_1 is isomorphic either to ℓ_1 or to L_1. Also open is whether there are more than $2^{\aleph_0}$ closed ideals in $L(L_1)$.

The new ideals are a familty $(\overline{\mathcal{I}}_{U_q})_{2<q<\infty}$, where $U_q : \ell_1 \to L_1\{-1, 1\}^{\mathbb{N}}$ maps the unit vector basis of ℓ_1 to a carefully chosen $\Lambda(q)$-set of characters. (A set of characters is $\Lambda(q)$ if the L_1 norm is equivalent to the L_q norm on their linear span.) Bourgain's solution to Rudin's $\Lambda(q)$-set problem is used.

The problem is to show that these ideals are all different.

(Large) Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

[S '75] There are infinitely many isomorphically different complemented subspaces of L_p, each isomorphic to its square, hence there are infinitely many (large) closed ideals in $L(L_p)$.

[Bourgain, Rosenthal, S '81] There are $\aleph_1$ isomorphically different complemented subspaces of L_p, each isomorphic to its square, hence there are $\aleph_1$ (large) closed ideals in $L(L_p)$.

This left open the question whether there are more than $\aleph_1$ (large?/small?) closed ideals in $L(L_p)$? Maybe there are even $2^{2^{\aleph_0}}$ (large?/small?) closed ideals.

The following solved the first problem for small ideals

Theorem. (Schlumprecht, Zsák '18)

There are infinitely many; in fact, at least $2^{\aleph_0}$; (small) closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$.

The ideals constructed in [SZ '18] are all of the form $\overline{\mathcal{I}}_U$ with U a basis to basis mapping from ℓ_r to ℓ_s but the bases for ℓ_r, ℓ_s are not the standard unit vector basis.

More Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

We proved,

Theorem. (Johnson, Schechtman 2021)

There are $2^{2^{\aleph_0}}$ closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$. Moreover, these ideals are even non-mutually isomorphic as Banach algebras.

We actually proved that there are $2^{2^{\aleph_0}}$ LARGE closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$.

and also that there are $2^{2^{\aleph_0}}$ small closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$.

The proof relays on fine properties of spaces spanned by independent random variables in L_p, $2 < p < \infty$, a topic investigated mostly by Rosenthal in the 1970-s.

Digression: Eidelheit

Before explaining more on the form of these new ideas, let me talk about a recent observation of Bill Johnson, Chris Phillips and myself that show that two closed ideals in $L(X)$ that are different are also not isomorphic as Banach algebras. This implies the “Moreover" statements in the theorems above.

Eidelheit (1940) proved that if $L(X)$ and $L(Y)$ are isomorphic as Banach algebras then X and Y are isomorphic as Banach spaces by an isomorphism U and the algebras isomorphism between $L(X)$ and $L(Y)$ is given by $A \to UAU^{-1}$.

Digression: Eidelheit

Digging into the proof we noticed that the same conclussion holds if a closed subalgebra of $L(X)$, containing the finite rank operators, is isomorphic as Banach algebra to a subalgebra of $L(Y)$, containing the finite rank operators.

If follows that if two closed ideals I and J of $L(X)$ are isomorphic as Banach algebras then, since they contain the finite rank operators, there is an isomorphism U of X onto itself such that $A \to UAU^{-1}$ maps I onto J. But since I is an ideal UAU^{-1} is in I. So $J \subset I$. Similarly, $I \subset J$. so $I = J$.

Now back to the constructions of ideals in $L(L_p)$:

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

Recall that for a sequence $u = \{u_j\}_{j=1}^{\infty}$ of positive real numbers and for $p > 2$, the Banach space $X_{p,u}$ is the real sequence space with norm

$$\|\{a_j\}_{j=1}^{\infty}\| = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

Rosenthal proved that $X_{p,u}$ is isomorphic to a complemented subspace of L_p with the isomorphism constant and the complementation constant depending only on p.

If u is such that $\lim_{j\to 0} u_j = 0$ but $\sum_{j=1}^{\infty} |u_j|^{\frac{2p}{p-2}} = \infty$ then one gets a space isomorphically different from ℓ_p, ℓ_2 and $\ell_p \oplus \ell_2$.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

$$\|\{a_j\}_{j=1}^{\infty}\|_{X_{p,u}} = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

However, for different u satisfying the two conditions above the different $X_{p,u}$ spaces are mutually isomorphic. We denote by X_p any of these spaces. We'll need more properties of the spaces $X_{p,u}$ but right now we only need the representation above and we think of $X_{p,u}$ as a subspace of $\ell_p \oplus_\infty \ell_2$.

Let $\{e_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_p and $\{f_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_2. Let $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ be two positive real sequences such that $\delta_j = w_j/v_j \to 0$ as $j \to \infty$. Set

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \quad \text{and} \quad g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Then $\{g_j^v\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,v}$ and $\{g_j^w\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,w}$.

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \text{ and } g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Define also $\Delta = \Delta(w, v)$

$$\Delta : X_{p,w} \to X_{p,v}$$

by

$$\Delta g_j^w = \delta_j g_j^v.$$

Note that Δ is the restriction to $X_{p,w}$ of

$$K : \ell_p \oplus_\infty \ell_2 \to \ell_p \oplus_\infty \ell_2$$

defined by

$$K(e_j) = \delta_j e_j \text{ and } K(f_j) = f_j$$

Consequently, $\|\Delta\| \leq \|K\| = \max\{1, \max_{1 \leq j < \infty} \delta_j\}$.

Rosenthal proved that X_p^* contains isoporphic copies of ℓ_r for all $q = p/(p-1) \leq r \leq 2$

A major technical part in our proof is the fact that Δ^* preserve non trivial copies of ℓ_r^n. More precisely,

For any sequence $r_i \nearrow 2$ and n_i such that $n_i^{\frac{1}{r_i}-\frac{1}{2}} \nearrow \infty$ (i.e. $d(\ell_{r_i}^{n_i}, \ell_2^{n_i}) \to \infty$) there are sequences $v = \{v_j\}_{j=1}^\infty$ and $w = \{w_j\}_{j=1}^\infty$ such that $\delta_j = w_j/v_j \to 0$ and

$\Delta^* = \Delta^*(w, v)$ isomorphically uniformly preserves these copies of $\ell_{r_i}^{n_i}$.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

For $1 < q < 2$, we construct new ideals of $L(X_p^*)$ of the form

$$\overline{\mathcal{I}}_{\Delta^*(w,v)},$$

that is the ideal of all operators factoring through $\Delta^*(w,v)$, for different sequences $(w,v) = \{w_i, v_i\}$.

More precisely, we build a continuum $\mathcal{C}$ of different sequences (w,v) such that $\overline{\mathcal{I}}_{\Delta^*(w,v)}$ are all different. This already produces a continuum of different ideals.

If $\mathcal{A} \subset \mathcal{C}$ one can look at the closed ideal generated by $\{\Delta^*(w,v)\}_{(w,v)\in\mathcal{A}}$ We show moreover that (with the right choice of $\mathcal{C}$) if $\mathcal{A} \neq \mathcal{B}$ then the two closed ideal generated by $\mathcal{A}$ and $\mathcal{B}$ are different.

This produces the required $2^{2^{\aleph_0}}$ ideals.

More Large Ideals in $L(L_p)$, main proposition

For appropriate (w, v) the operator $T = \Delta^*(w, v)$ has the following properties:

X (in our case $X^*_{p,v}$) is a Banach space with a 1-unconditional basis $\{e_i\}$ (i.e., the norm of a linear combination depends only on the absolute value of the coefficients). $T : X \to X$ is a norm one operator satisfying:

(a) For every M there is a finite dimensional subspace E of X such that $d(E) := d(E, \ell_2^{\dim E}) > M$ and $\|Tx\| \geq 1/2$ for all $x \in E$.

and
(b) For every m there is an n such that every m-dimensional subspace E of $[e_i]_{i \geq n}$ satisfies $\gamma_2(T_{|E}) \leq 2$.

Here, for $T : X \to Y$,

$$\gamma_2(T) = \inf\{\|S\|\|R\|;\ T = SR, R : X \to H, S : H \to Y\}.$$

Main proposition

Proposition

Let $T : X = [e_i] \to X$ *satisfy (a) and (b). Then there exist a subsequence of* $\mathbb{N}$, $1 = p_1 < q_1 < p_2 < q_2 < \dots$ *with the following properties:*
Denoting for each k, $G_k = [e_i]_{i=p_k}^{q_k}$. *Let* $\mathcal{C}$ *be a continuum of subsequences of* $\mathbb{N}$ *each two of which has a finite intersection.*
For each $\alpha \in \mathcal{C}$, $P_\alpha : X \to [G_k]_{k\in\alpha}$ *denotes the natural basis projection and* $T_\alpha = TP_\alpha$.
If $\alpha_1, \dots, \alpha_s \in \mathcal{C}$ *(possibly with repetitions) and* $\alpha \in \mathcal{C} \setminus \{\alpha_1, \dots, \alpha_s\}$ *then for all* $A_1, \dots, A_s \in L(X)$ *and all* $B_1, \dots, B_s \in L(X)$

$$\|T_\alpha - \sum_{i=1}^{s} A_i T_{\alpha_i} B_i\| \geq 1/4.$$

Main proposition

Corollary.

Let $T : X = [e_i] \to X$ satisfy (a) and (b). Then $L(X)$ contains $2^{2^{\aleph_0}}$ different closed ideals.

In particular, for appropriate $r_i \nearrow 2$ and $n_i \to \infty$, $X = (\sum \oplus \ell_{r_i}^{n_i})_2$ and thus also $X = (\sum \oplus \ell_{r_i})_2$ satisfy that $L(X)$ contains $2^{2^{\aleph_0}}$ different closed ideals.

Unfortunately these are not subspaces of L_p but using the Corollary with the operator $T = \Delta^*(w, v)$ discussed above proves the theorem.

Freeman, Schlumprecht, Zsák

Recently, Freeman, Schlumprecht and Zsák found a different criterion, of a similar nature, for a Banach space (with unconditional FDD) to have $2^{2^{\aleph_0}}$ closed ideals

Using it they proved that for any $1 < p < q < \infty$ $\ell_p \oplus \ell_q$ has $2^{2^{\aleph_0}}$ closed ideals. Same is true also for $\ell_1 \oplus \ell_p$ and $\ell_p \oplus c_0$.

Since $\ell_p \oplus \ell_2$ is isomorphic to a complemented subspace of L_p, this gives another proof of our result (for small ideals).

Problem: (FSZ) Is there a separable Banach space X such that $L(X)$ has exactly countably many closed ideals?

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