The Work of Caucher Birkar

Allyn Jackson

Caucher Birkar is a mathematician of great originality and depth. His research area, algebraic geometry, addresses fundamental questions about the nature of abstract geometric spaces. These questions can often be stated quite simply, but their resolution requires the use of a massive technical apparatus. Not only has Birkar completely mastered this apparatus, he also possesses profound geometric intuition that allows him to go beyond technical achievements and break new conceptual ground. His work has produced major advances in birational geometry, in particular in a central paradigm known as the Minimal Model Program.

In algebraic geometry, the basic object of study, called an algebraic variety, is the solution set of a collection of polynomial equations. The solution set can take different forms depending on the range of the variables in the equations. Consider for example the equation $x^2 + y^2 = z^2$. If x, y, and z range over the integers, then the solution set is the collection of Pythagorean triples. If x, y, and z range over the real numbers, then the solution set is a cone in three-dimensional space. If x, y, and z are complex numbers, then the solution set is not possible to visualize directly; it is an abstract space that inherits geometric structure from the complex numbers.

Because this geometric structure brings richness and flexibility, algebraic geometers usually concentrate on complex algebraic varieties. These varieties exhibit great diversity, but one can also perceive similarities that some varieties share. The main goal of birational geometry is to provide a good way to classify all complex algebraic varieties up to birational equivalence. In pursuit of this goal, the Minimal Model Program (MMP) proposes a way to identify special varieties in each class that are in some sense the simplest and that provide building blocks out of which one can construct other, more complicated varieties.

The roots of birational classification can be traced to the work of the great 19th century geometer Bernhard Riemann, who studied the case of one-dimensional complex varieties. To any such variety one can associate a Riemann surface, which is a real two-dimensional surface with extra geometric structure inherited from the complex numbers. Such surfaces come in three different types: those with positive curvature, like a sphere; those with zero curvature, like a donut with one hole; and those with negative

curvature, like a donut with several holes. The number of holes provides a natural invariant for the classification of one-dimensional varieties.

Two-dimensional varieties were the subject of a great deal of research by Italian algebraic geometers working around the early 20th century. They used extensively the concept of *birationality*: When two varieties are birationally equivalent, they are essentially the same, apart from some small subsets that can safely be ignored. Birational equivalence provides a flexible way to classify varieties. The Italian geometers discovered that one can make a two-dimensional complex variety more complicated by "blowing up," or expanding, a point into a special kind of curve called a (-1)-curve. One can also reverse the procedure and simplify the variety by "blowing down", or contracting, a (-1)-curve to a point. Blowing up or blowing down a variety results in a new variety in the same birational equivalence class. Repeating the blowing-down procedure as many times as possible results in an especially simple variety.

Echoing the picture in the one-dimensional case, these simple two-dimensional varieties fall into three categories. The first, today called Mori-Fano fiber spaces, are built out of Fano varieties; these varieties are a natural generalization of a Riemann sphere, which has positive curvature. The second category, called Calabi-Yau fiber spaces, are built out of Calabi-Yau varieties; these are a natural generalization of a donut shape, which has zero curvature. The third category, called varieties of general type, are a natural generalization of a Riemann surface having negative curvature, that is, a surface with at least 2 holes.

The methods developed to treat two-dimensional varieties could not resolve the three-dimensional case, and a new approach was needed. This came in the 1970s and 1980s, in the work of Shigefumi Mori. Because (-1)-curves do not exist in three dimensions, he had to develop an entirely new blowing-down procedure. This new procedure produced singularities, that is, points on the variety that are not smooth. Then-new advances in singularity theory treated some cases, but others required the use of a new tool called a flip. Intuitively, this is a process wherein one slices out a region, flips the region around, and pastes it back in. Mori's proof that flips exist in three dimensions was the key to establishing the MMP as a way to classify varieties. His work earned him a Fields Medal in 1990.

At the coarsest level, the classification of three-dimensional varieties exhibits the same motif as in the two dimensional case, in that the especially simple varieties fall into three categories: Fano-Mori fiber spaces, Calabi-Yau

fiber spaces, and varieties of general type. However, the finer classification within each grouping is much more complicated in three dimensions than in two because three-dimensional varieties are inherently more complicated than two-dimensional ones. Varieties of the Calabi-Yau fiber spaces type and varieties of general type are called "minimal models" in the MMP terminology.

During the 1980s and 1990s, several mathematicians, including Vyacheslav V. Shokurov, developed a more general form of the MMP called log MMP, in which each variety is paired with a collection of varieties of one dimension less. It turns out that studying these pairs adds a powerful kind of flexibility that can be exploited in many proofs. These developments ignited hope that, despite daunting technical challenges, the MMP could be extended to dimensions higher than three. Among the toughest challenges were how to handle singularities and how to prove the existence of flips in higher dimensions. The latter became an outstanding open problem in algebraic geometry.

A major step forward came in 2003 in a paper by Shokurov, which expanded on some of his earlier work on flips in dimension three, to establish existence of flips in dimension four. More important than this specific result was the new philosophy Shokurov outlined for addressing the MMP in higher dimensions. In contrast to Mori's concrete, geometric approach, Shokurov's work was much more algebraic, drawing on ideas from highly abstract branches of mathematics such as cohomology theory.

As a doctoral student of Shokurov, Caucher Birkar absorbed and refined the new philosophy while also mastering the formidable technical machinery required. In 2006, two years after completing his doctorate, Birkar produced a major breakthrough in collaboration with three other mathematicians. The paper they wrote is now universally known as BCHM, for the last names of the four authors: Caucher Birkar, Paolo Cascini, Christopher Hacon, and James McKernan.

One of the main results in BCHM concerns the canonical bundle, which is a construct that stands at the center of birational geometry. Defined at any point on a variety, the canonical bundle encapsulates in an especially useful way a great deal of geometric information about the variety. By taking the sections of the canonical bundle and its powers, one produces an algebraic object called the canonical ring. BCHM answered affirmatively the longstanding question of whether the canonical ring is finitely generated. A local version of this result, also proved in BCHM, allowed the authors to establish

the existence of flips in all dimensions greater than two. They then were able to prove the existence of minimal models for varieties of general type.

BCHM changed the landscape of research in birational geometry, opening areas previously thought to be inaccessible. The tools and perspectives introduced in the paper have become widely used and have had an enormous impact. Still, many mysteries remained, especially concerning varieties other than those of general type.

So it was with great excitement that the mathematical world greeted two papers by Birkar in 2016, which focus on the case of Fano varieties. The pinnacle of this tour de force is Birkar's proof of the Borisov-Alexeev-Borisov conjecture, which predicted that, under reasonable assumptions, Fano varieties form a bounded family. More specifically, Birkar showed that, in any fixed dimension, Fano varieties with mild singularities can be indexed using a finite number of parameters. Birkar's papers also contain spectacular advances in treating certain types of singularities that arise in the MMP. This work is expected to lead before long to the solution of many outstanding problems in birational geometry. Seminars and workshops have been held worldwide in which mathematicians studied and absorbed Birkar's work.

The MMP has yet to be fully established in all dimensions. In particular, the case of higher-dimensional varieties that are predicted to have Calabi-Yau fiber spaces as minimal models remains mysterious. Birkar's work is sure to guide and inspire new developments.

The MMP as discussed so far pertains to varieties for which the variables in the defining polynomials range over the complex numbers. A new frontier is the case where the variables range over other sets of numbers. For example, given the set $\{0, 1, 2, 3, ...p - 1\}$, where p is a prime number, one can define arithmetic operations to be addition and multiplication mod p (this is analogous to "clock arithmetic"). This set, together with these operations, is called a field of characteristic p. Given a set of polynomials, one can allow the variables to range over a field of characteristic p.

Varieties in characteristic p are of great interest because they have many connections to number theory, so their birational classification would be very useful. That classification is expected to bear some resemblance to that over the complex numbers. However, many of the tools developed for the complex case simply do not apply in characteristic p, so an entirely new approach is needed. Birkar has delved into this burgeoning area, making several significant contributions. In particular, building on work of Christopher Hacon and Chenyang Xu, Birkar proved the existence of log flips and log minimal

models for three-dimensional varieties over a field of characteristic p when $p > 5$.

With a taste for deep, fundamental problems that stand at the heart of algebraic geometry, Birkar has emerged as a new leader in the field. Not only does he possess great facility with many of the latest tools at the frontier of this highly technical subject, he has also created some of the most powerful innovations. He brings to this work a strong geometric intuition as well as a fearlessness in tackling longstanding, difficult problems. Caucher Birkar stands poised to make yet more outstanding contributions to mathematics.