The Work of Akshay Venkatesh

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The study of numbers has always been at the heart of mathematics. While an enormous body of knowledge has accumulated over the centuries, number theory still retains endless mysteries that grow out of the simplest concepts: Relations among the integers. Because the integers comprise the bedrock out of which all of mathematics grows, number theory has connections to many other branches of the field. Number theorists draw on ideas from analysis, algebra, combinatorics, and geometry, as well as from other fields like theoretical physics or computer science.

Even in a subject requiring such breadth, Akshay Venkatesh stands out for the startlingly original way he has connected number theory problems to deep results in other areas. Far from using them as "black boxes" to crank out solutions, Venkatesh brings fresh insights to the results and highlights their unexpected connections to number theory. In this way he has made striking advances in number theory while also greatly enriching other branches of mathematics.

Because the oeuvre of Akshay Venkatesh is so diverse, a complete overview is not possible in a short space. What follows therefore is a description of three examples of the work of Venkatesh and his co-authors that exemplify the depth and wide-ranging nature of his work.

The starting point for the first example is polynomial expressions such as $x^2+xy+7y^2+yz+12z^2$. These expressions encapsulate tremendous intricacy because they model the complex interactions between addition and multiplication. The expression above is an example of a quadratic form, which is a polynomial in any number of variables where the highest exponential power is 2. A basic question is which integers are produced by substituting integral values for the variables in a quadratic form. For example, the quadratic form x^2 produces only perfect squares, while $w^2 + x^2 + y^2 + z^2$ produces *all* the integers, a fact proved in 1770 by Joseph-Louis Lagrange.

In his monumental *Disquisitiones Arithmeticae*, published in 1801, Carl Friedrich Gauss showed how to transform one quadratic form into another by simple substitutions of variables. Such a transformation can greatly simplify the quadratic form, and any integer produced by the simpler form is also produced by the original one (though the converse need not be true). If we can in this way transform a quadratic form P in m variables into Q in n

variables, with $m \geq n$, we say that P *represents* Q. When is one quadratic form represented by another? This question is a variant of the 11th problem on David Hilbert's famous problem list from 1900. In 1978, John S. Hsia, Yoshiyuki Kitaoka, and Martin Kneser reached a major milestone by proving that P represents Q if $m \geq 2n + 3$ (there are some obvious exceptions that need not concern us).

For thirty years, mathematicians did not know whether a more-precise relationship between m and n could be established. The received wisdom was that the best result would probably be $m \geq 2n+2$. This was why it came as such a surprise when, in 2008, Venkatesh and his co-author Jordan Ellenberg showed that m and n could be much closer than previously thought, namely, that P represents Q if $m \geq n + 5$.

Even more surprising was their method of proof, which made use of powerful insights from dynamical systems theory. They started by viewing the problem in the context of a lattice, which is a regularly-spaced set of points in n-dimensional space, thought of as marking integer points. The use of lattices in number theory has a long tradition, going back to the "geometry of numbers" concept of Hermann Minkowski in the late 19th century. Lattices are also a natural domain for dynamical systems, where one can think of the lattice points flowing under the influence of a system changing over time, like dust particles flowing in a breeze.

Some of the deepest insights about flows on lattices are found in a landmark theorem proved by Marina Ratner in the early 1990s. Her theorem predicts where lattice points move under the long-term influence of a dynamical system. Ellenberg and Venkatesh utilized a variant of Ratner's theorem that allowed them to apply lattice dynamics to the problem of representation of quadratic forms. An outstanding achievement in number theory, this work also led to much excitement among researchers in dynamical systems, who quickly added the new insights to their toolbox.

The work on representation of quadratic forms is related to powerful results by Venkatesh that appeared in preprint form in 2005 (published in 2010 and later expanded in joint work with Philippe Michel). Here Venkatesh explored a more-technical question known as the subconvexity problem, putting it into a totally new context that highlighted connections to other mathematical areas, particularly dynamical systems, in a bold and innovative way.

The second example of the work of Venkatesh exhibits similar virtuosity in producing surprising advances in number theory by bringing in tools from yet another area, this time topology. A basic characteristic of the integers is unique factorization: Any integer can be expressed, in exactly one way, as a product of prime numbers. The integers form the quintessential example of a *ring*, which is a set of objects together with three operations analogous to addition, subtraction, and multiplication. One example of a ring, call it R, is the set of numbers like $a + b\sqrt{-5}$, where a and b are integers. R has the virtue that it contains a solution to an equation like $x^2 = -5$, which is not solvable in the integers.

But R has a shortcoming: It lacks unique factorization. For example, in R one can factorize 9 in two ways, as 3×3 and as $(2 + \sqrt{-5}) \times (2 - \sqrt{-5})$. The *class number* of a ring is an integer that measures how badly the ring fails at unique factorization. Class numbers crop up all across number theory. Exactly how they vary over the collection of all rings has remained a mystery.

With few theoretical tools available to understand class numbers, mathematicians resorted to simply computing class numbers and searching for patterns. It was through such experimentation, combined with far-reaching insight, that Henri Cohen and Hendrik Lenstra in 1984 formulated some surprising heuristic results. For example, one would reasonably expect that class numbers are randomly distributed across the integers, so that about one-third of them would be divisible by 3. Astonishingly, the Cohen-Lenstra heuristics predict that the proportion is actually around 43%.

Why do the Cohen-Lenstra heuristics work? The answer remained elusive, despite much research over the past 30 years. Then a big breakthrough came in 2016, in work by Venkatesh together with co-authors Jordan Ellenberg and Craig Westerland. Focusing on a narrower problem of class numbers not of rings of integers but of rings in related objects known as function fields, they made a major advance in understanding the Cohen-Lenstra heuristics. Their strategy was to translate the question into the realm of topology, a branch of mathematics that studies basic properties of shapes.

Mathematicians already knew that function-field problems could be translated into topology. But topology is a big field with a vast toolkit. Which tools would apply? This was where the insight of Venkatesh proved crucial, leading him and his co-authors to exploit the topological notion of *homological stability*, a very modern concept that has been the subject of a great deal of recent research. Venkatesh and his co-authors proved a new result in topology, the homological stability for a class of topological objects known as Hurwitz spaces, and then translated this result back to the realm of number theory to prove the validity of some of the Cohen-Lenstra predictions for function fields. Once again the work of Venkatesh enriched two widely separated branches of mathematics.

The final example of the work of Venkatesh is not a finished result, but rather a set of bold new conjectures he and his co-authors have formulated. Seeking to explain profound connections between phenomena in topology on the one hand and number theory on the other, these conjectures show Venkatesh as a trailblazer of new directions in research. The ideas spawned much excitement in seminars he led during the 2017-2018 academic year.

This conjectural work relates to the Langlands Program, which today drives a great deal of mathematical research. The Langlands Program envisions a web of relationships among a variety of phenomena arising in different branches of mathematics, including topology, analysis, algebra, and number theory. Mathematicians are a long way from fulfilling the whole of the Langlands Program, but some special cases have been affirmed. Perhaps the best known example is the proof in the 1990s of Fermat's Last Theorem, carried out by Andrew Wiles with crucial input from Richard Taylor.

The Taylor-Wiles method that emerged from that work has become a powerful tool for connecting geometric objects known as elliptic curves to analytic objects known as modular forms—these are exactly the kind of connections predicted by the Langlands Program. As powerful as it is, the Taylor-Wiles method, as originally developed, applied only in a restricted setting, namely that of special geometric objects called Shimura varieties. Recent results generalizing the Taylor-Wiles method to non-Shimura varieties figure prominently in the newest work of Venkatesh.

This work centers on a set of topological objects known as locally symmetric spaces. Venkatesh and his co-authors have found that the topology of these spaces harbor unexpected symmetries. These symmetries occur in the homology groups of locally symmetric spaces; the homology group of a space can be loosely thought of as measuring the holes in the space. Venkatesh has formulated a vision for explaining these symmetries by appealing to a different mathematical area known as motivic cohomology. The explanation uses the generalized Taylor-Wiles method and, as a byproduct, might yield deeper insights into that method. The work is far from complete, but early expectations are that it will provide a key step in the ascent towards a full understanding of the Langlands Program.

Most mathematicians are either problem-solvers or theory-builders. Akshay Venkatesh is both. What is more, he is a number theorist who has developed an unusually deep understanding of several areas that are very different from number theory. This breadth of knowledge allows him to situate number theory problems in new contexts that provide just the right setting to highlight the true nature of the problems. Only 36 years of age, Venkatesh will continue to be an outstanding leader in mathematics for years to come.