David Donoho

Citation: "The Gauss Prize is awarded for his fundamental contributions to the mathematical, statistical and computational analysis of signal processing."

Magnetic Resonance Imaging scans (MRIs) are crucial to high-tech medical care: Their three-dimensional view inside your body allows doctors to spot an aneurysm on the edge of bursting, to fly through your brain to plan a surgery, or to pinpoint a hairline crack in a bone — all with nary a scalpel or dose of radiation. However, when you are the patient getting scanned, MRI scans impose very low-tech demands. You are asked to lie perfectly still for as much as an hour, inside a cramped tube that clangs and thumps.

But now the seemingly endless scanning process is about to go by ten times faster, thanks to a new generation of MRI scanners now entering clinical use. The new technology can save time and money on the 80 million MRIs performed each year globally, and will make MRIs practical for fidgety children who can't stay still for long scans. The speed-up also enables ambitious 3D scans and MRI "movies" of the beating heart.

The engineers and doctors who brought these new devices to market were inspired by insights crystallized in mathematics journals in 2006. Those insights today go by the name "compressed sensing," a term coined by the 2018 Gauss Prize winner David Donoho. The Gauss Prize recognizes mathematical work with impacts beyond mathematics.

Along with Emmanuel Candès, Terence Tao and other mathematicians, Donoho published mathematical analyses in the mid-2000's showing that compressed sensing (CS) might work practically, and proposed algorithms that were successful enough to inspire further research. A massive outpouring of mathematical and experimental work soon followed, with applied mathematicians, harmonic analysts, and information theorists pushing the theory; numerical analysts and computer scientists creating fast algorithms to enable computation; and MRI researchers adding their own profound understanding of MR physics and many additional creative insights.

FDA approval of new medical devices sets a high bar for any would-be innovation. Yet FDA approvals for compressed-sensing devices were reached in 2017, barely a decade after initial appearance of CS articles in mathematics journals. That's impact!

I. Sparsity.

The 2018 Gauss Prize winner Donoho was one of the first researchers to develop math describing signals that are *sparse*. Such signals are zero most of the time, with occasional non-zero wiggles. Examples are all around you. Think of the night sky: an occasional star (represented, say, by a very lonely "1") punctuates the vast blackness (represented by a sea of "0"). Think of the human genome: two people differ only once every 300 nucleotides.

Donoho first encountered sparsity just after university, while working in oil exploration. To find oil deep underground, geophysicists would set off explosions, sending seismic waves into the earth. Each time the wave hit a new rock layer, it sent back an echo; from the echo signal, the scientists could reconstruct an image of the layers below. The seismic echo series was sparse because layer changes were relatively rare.

II. L_1-norm

At age 21, Donoho stumbled on a puzzle that marked his career. In those days, raw seismic measurements only offered a vague, blurry sense of where the rock layers were. But geophysicists had developed methods that seemed amazingly effective at "deblurring" the signal and identifying the layer changes precisely.

Those methods measured distance in a nontraditional way. Ever since foundational work by the mathematical giant C.F. Gauss – as in Gauss Prize -- scientists traditionally use the so-called L_2 distance in data processing. This distance is also called the crow's-flight or Euclidean distance because it measures the length of a path that goes straight between two points, like in high school geometry. However, the surprising new methods used instead the "L_1 norm", also called the Manhattan distance, because it measures how many city blocks you would walk if you have to travel between two points on a rectangular grid of city streets (diagonals like Broadway not allowed!).

There was something mysteriously effective in this combination of sparse signals with L_1 norm — but no one knew why. When Donoho returned to graduate school for his Ph.D., he was determined to solve the puzzle. In the coming years, he developed mathematical theory showing the unreasonable effectiveness of the L_1 norm with sparse signals.

Some of the phenomena seemed miraculous. He first used L_1 + sparsity techniques to recover a sparse signal that has been blurred in an unknown, arbitrary way (today called `blind deconvolution'). He next used them to recover totally missing data. Often in signal processing, some part of a signal can go missing – think of an old acoustic recording with no highs or lows. Donoho, Philip Stark and Ben Logan showed that for certain special signals — sparse ones — L_1 + sparsity techniques could *perfectly* recover missing low-frequency signal. In other work, Donoho and collaborators Jeffrey Hoch and Alan Stern developed L_1 + sparsity techniques to recover missing high-frequencies – in acoustic terms, missing `high notes'.

L_1 + sparsity also allowed `de-noising' of signals: if you add noise to a sparse signal and then look at the plot, you will see `daisies' -- signal -- sticking up above `weeds' – noise; L_1 minimization gives a way of chopping out the weeds while keeping the daisies. Donoho and Iain Johnstone showed that for sparse signals this was essentially optimal.

III.   Harmonic Analysis

The 80's/90's `wavelet revolution' in applied mathematics further transformed Donoho's thinking. At the time, computational harmonic analysts such as 2014 Gauss Prize Winner Yves Meyer and collaborators Ronald Coifman, Ingrid Daubechies, and Stephane Mallat were building many new tools for mapping digital signals into more useful forms. Their new wavelet transforms literally blew Donoho's mind. Transforming digital data using these new tools revealed that sparsity was *everywhere* -- in images and other media we now use daily. To Donoho's mind, this dramatically expanded the stage for applications of L_1+Sparsity.

Donoho's mathematical results placed a premium on being as sparse as possible. They drove him to `sparsify' even better than wavelets could – where possible.   He searched for systems `beyond wavelets' that would expose the hidden sparsity of geometric phenomena such as edges, sheets, and filaments in images. His collaborators Emmanuel Candès and Jean-Luc Starck were soon also aiming beyond wavelets, for "curvelets", "beamlets", and other `X'-lets.

Donoho worked to sparsify signals even more by combining several different systems of harmonic analysis. For example, a sine wave contaminated with several spikes would not be sparse under traditional Fourier analysis, but it could be sparsely synthesized using both Fourier analysis and wavelets together. With collaborators Michael Saunders and Scott Chen, he developed an algorithm called Basis Pursuit to solve the synthesis problem by minimizing the L_1 norm. Its success seemed miraculous because the task -- solving a system of underdetermined equations, and algorithmically getting the sparsest possible answer – seemed forbiddingly complex. With collaborators Xiaoming Huo, Michael Elad and Vladimir Temlyakov, Donoho gave a series of foundational mathematical results showing that the L_1 norm could truly find the sparsest possible such synthesis.

IV.   Compressed Sensing

The 3 strands of research described in Sections I-III above converged in the mid-2000's to produce Compressed Sensing, the mathematical theory that inspired those fast MRI's that are now coming to market. The sparsity of images when viewed in the wavelet basis; the use of the L_1 norm; the use of underdetermined equations – all three ingredients came together in work by Donoho and by Candès, Romberg, and Tao, mentioned earlier. Their mathematical analyses showed clearly why all three ingredients must be present to allow speed ups, and how, under certain assumptions, this combination is guaranteed to work. Such clear mathematical understanding was transformational, and inspired rapid progress, in MRI research, and elsewhere.

V.   Unreasonable Effectiveness

Since his university days, Donoho has believed that mathematicians would contribute during the information era by providing new models for data, new processing algorithms, and subtle but powerful theoretical insights. And he himself has done all three.

What Donoho did not know as a youngster, and could not have known, is that the continuing growth of pure mathematics would be so important. For example, in his own work on compressed sensing, Donoho found that essential roles were played by the theories of random matrices, of high-dimensional Banach spaces, of random convex polytopes, and of mathematical spin glasses – in all cases, pure mathematics unrelated to signal processing and much younger than Donoho himself!

50 years ago, Eugene Wigner coined the phrase `unreasonable effectiveness of mathematics' to refer to the surprising tendency of pure mathematics to inspire practical applications.

If, some day, you enjoy a fast MRI scan, you may perhaps also remember Wigner's dictum!