Subhash Khot

Typically, major math prizes are given for major results. But in this case, Subhash Khot is receiving the Nevanlinna Prize in large part for a conjecture – and even more surprisingly, one whose truth experts can't yet decide on.

But Khot's Unique Games Conjecture has already amply proven its value, even should it ultimately be disproven. It has cast a bright light on previously dim areas of computational complexity and provided critical insight — and, yes, Khot has also used it to prove major results, ones that stand regardless of its truth.

The conjecture has opened up a particularly fruitful way of addressing the central question of the field of computational complexity: How hard are problems to solve? More precisely, if you found the cleverest possible way to solve a particular problem, how quickly could a computer find the answer using it?

Computer scientists are nearly certain that some problems are so difficult that computers can't reliably find the answer at all, at least not in any reasonable amount of time (such as before the universe ends). That's the famous conjecture known as $P \neq NP$, and it has resisted proof for four decades – though computer scientists have only become more convinced that it must be true over time.

So many researchers have moved on to the next question: If a problem is too hard for a computer to solve quickly and precisely, can it at least find a good approximation? In the real world, after all, a good approximation is usually enough.

Before Khot's work, researchers had found a few problems for which the answer was no, but for most problems, they had little idea. Khot found a remarkably simple problem (called Unique Games) that seems to encapsulate what makes many problems hard to solve even approximately in a reasonable amount of time. His conjecture is that it's not just hard but impossible to reliably find an approximate answer to Unique Games reasonably quickly. In a certain precise, technical sense, Unique Games seems to be the simplest really, really hard problem.

Whether he's right or wrong, his problem has "cleaved nature at its joints," as the early taxonomist Carl Linnaeus put it. The conjecture is proving to be a kind of lever point, a spot where applying effort yields big results. Assuming the conjecture is true, Khot and others have shown that the vast majority of problems computer scientists care about also can't be approximated. Not only that, but the conjecture has shed light on seemingly unrelated problems in geometry, Fourier analysis, and even the mathematics of foams and voting, and those results don't rely on its truth.

The Unique Games problem is an elaboration of one that a six-year-old could play with. Imagine that you have a box of crayons and a drawing of a bunch of bubbles, some of which have lines connecting them. (Computer scientists call such drawings "networks.")

Can you find an efficient way to color in the bubbles so that any two connected ones are different colors?

If your box has only two crayons (say yellow and purple), you can figure this out quite efficiently. Start with an arbitrary bubble and color it yellow. Since all the bubbles connected to it now have to be purple, color them in. Continue this way until you've either managed to color in the whole network or you've found a bubble that's connected to both yellow and purple bubbles, making the project impossible.

If you add just one more crayon, though, this method fails, because when you color the first bubble yellow, you don't know what color the connected ones have to be. So if you get to a bubble you can't color in without breaking the rules, you don't know if a different selection earlier would have solved the problem. The difficulty isn't just with that method – no other method will reliably and efficiently solve the problem either. It been proven to be NP-hard — in other words, effectively impossible.

Khot altered this problem slightly. He made it easier than the ordinary three-crayon problem by providing a rule so that whenever any bubble is colored in, the color of all connected bubbles is fixed. Then an algorithm like the earlier one applies, and it's easy enough to determine if the network can be colored in without breaking the rules. But for networks that are duds — ones that can't be colored in — Khot asked this: Which coloring breaks the fewest rules possible?

The Unique Games Conjecture (UGC) is that if you have lots of colors, you'll never find an efficient method to color in the drawing that's anywhere close to the best one no matter how clever you are.

Khot developed the conjecture in 2001 (in a slightly different formulation, which the name was derived from). A couple of years later, computer scientists got the first glimmer of its importance, when Khot and others found that if the UGC was true, then they could find firm limits on how well many other problems could be approximated.

Here's one: Imagine at a celebrity-studded party, guests sought out the stars to shake hands with them (but non-celebrity partygoers kept their hands to themselves with one another). If you know only who shook hands with whom, can you figure out the minimum number of celebrities that might have been at the party? A simple algorithm can find an approximate solution for this, but it might specify as much as twice as many celebrities as are really needed. Computer scientists long imagined you could do better with a more sophisticated algorithm, but in 2003, Khot, together with Oded Regev, showed that if the UGC is true, they were wrong. That simple algorithm is the best you can do.

Khot's next success came in 2005. Together with Ryan O'Donnell, Elchanan Mossel and Guy Kindler, Khot found that the UGC implied a similar limit for approximating solutions to a problem called "Max Cut," which asks you to take a network and split it into two groups so that the maximum number of edges passes between them. Then in

2008, Prasad Raghavendra showed that if the UGC is true, a very simple method can find the best approximations for an enormous class of problems called "constraint satisfaction problems." With this, computer scientists know exactly how well nearly any problem can be approximated.

Of course, the caveat is that all these results depend on the UGC being true. If it turns out to be false, the entire sparkling, beautiful theory is a mirage.

But the conjecture has proved remarkably powerful independent of its truth. In the process of using the UGC to discover how well other problems could be approximated, Khot and others have proven several significant theorems in other areas, including geometry and Fourier analysis.

These implications have even ranged as far as voting theory. Once all the votes in an election are cast, there are a variety of ways of determining the winner. One of the most obvious is that the election goes to the candidate with the majority of votes, but there are other choices too, such as the American electoral college system. Khot and his co-authors used intuition from the UGC to propose that majority-rules is the method for counting votes in which a few miscounted votes is least likely to change the election result. This was indeed confirmed later by others.

Another group was working on proving that the UGC was true, and while their method failed, the work led them to another discovery: They found a shape that in a certain sense lies halfway between a square and a circle (though in many more than two dimensions). Like a square, copies can be placed next to each other horizontally and vertically to fill a whole space without gaps or overlaps, forming a multidimensional foam. But its perimeter is much smaller than a square – it's closer to that of a circle, the object which has the smallest perimeter for the area contained.

In the meantime, other researchers have worked to prove that the UGC is wrong, with equal lack of direct success, but with equal collateral benefits. Although they haven't yet succeeded in finding an algorithm that can efficiently find a good approximate solution to Unique Games, they have developed some excellent new algorithmic methods for other circumstances.

Efforts to prove the conjecture, to disprove it, and to discover its consequences have all proven enormously fruitful. The Unique Games Conjecture will be driving research in theoretical computer science for many years to come.