Results and open problems in mathematical General Relativity

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1 General Relativity.

In contradistinction with the Newton's absolute space and time, and in generalization of Special Relativity, a spacetime of General Relativity is any differentiable manifold V endowed with a pseudo Riemannian metric g of Lorentzian signature. We take it to be $(\;\;, +, \ldots, +)$. A tangent vector X to V is timelike if $g(X,X) < 0$, null if $g(X,X) \quad 0$, causal if $g(X,X) \leq 0$. The manifold is usually of dimension four, but higher dimensions are being considered in the aim of unification of gravitation with the other fundamental forces of nature, electromagnetism, weak and strong interactions. The physical time marked by a clock as elapsed between two events is the g length of the timelike trajectory followed by the clock.

5 Cosmic censorship conjectures

A new phenomenon, largely verified by observation today, is the existence of black holes, predicted by the Schwarzschild solution of the Einstein equations which models the gravitational field exterior to a spherically symmetric body

$$-(1-\frac{2m}{r})dt^2+(1-\frac{2m}{r})^{-1}dr^2+r^2dS^2. \tag{5.1}$$

Though the metric can be extended to a smooth metric for $r > 0$, no causal line can escape from the interior $r < 2m$, the boundary $r = 2m$ is null (a characteristic hypersurface called horizon). The singularity at $r = 0$ cannot be seen by a far away observer. Penrose has formulated the conjecture (weak cosmic censorship) that the non existence of what he called "naked singularities" is a generic properties of physically realistic Einsteinian spacetimes.

The strong cosmic censorship conjecture (which in fact does not imply the weak one) also originates from Penrose. It says that the globally hyperbolic Einsteinian development of complete, generic, physically realistic initial data is inextendible, even as a Lorentzian manifold. The maximal globally hyperbolic Taub spacetime, $S^3 \times (-\pi, \pi)$, extendible to $S^3 \times R$, is a counter example to the strong cosmic censorship, but it is non generic, due to its symmetry group.

The validity of the cosmic censorship conjectures has been studied in detail for Einstein equations with source a scalar field, by Christodoulou in the spherically symmetric case, with M R^3. He has shown that for small initial data, i.e. g near to the Euclidean metric, K near to zero as well as the scalar field initial data, the globally hyperbolic Einsteinian development is smooth and complete. For large initial data Christodoulou shows the existence of a global generalised solution, and proves that it may develop naked central singularities, but these singularities are unstable under small variations of the data.

6 Results for global existence (Vacuum case, n=3).

Asymptotically Euclidean space manifold.

Christodoulou and Klainerman proved (1993) the existence of future and past complete vacuum Einsteinian developments with initial data on R^3 near to the Minkowskian initial data. A shorter proof, using wave coordinates, has been given by Lindblad and Rodnianski 2004. Previous results by H. Friedrich 1988 proved the existence of future complete developments of initial data on an asymptotically null hyperboloid, using his conformal formulation of the Einstein equations. These results show that a conjecture of Einstein, that Minkowski spacetime was the only asymptotically Euclidean complete solution of the vacuum Einstein equations, had to be reformulated through the positive mass theorem.

Compact space manifold.

It is conjectured, and partially proved, that the manifold $T^3 \times R$ with the flat metric is the only future and past complete Einsteinian spacetime; it is expected that there will, generically, exist a singularity in one time direction, physically in a finite past, the big bang.

Various possibilities exist for the future.

The future complete existence has been proved in the following vacuum cases:

-Space T^3 with T^2 symmetry, without smallness assumption on the initial data (Berger, Chrusciel, Isenberg, Moncrief 1997)

-Space M a principal fiber bundle $M \to \Sigma$ with group and typical fiber $U(1)$ and basis Σ, a 2-manifold with genus grater than 1, with 1 parameter spacelike isometry group defined by the action of $U(1)$. The initial data are supposed to be near initial data of a flat spacetime (polarized case, i.e. orbits orthogonal trajectories to surfaces $\Sigma \times \{t\}$, C.B.-Moncrief 2001, general case, C.B 2004).

- Initial data near initial data obtained by quotients with discrete isometries of a cone of Minkowski space time (Andersson - Moncrief 2004).

7 S^1 symmetric vacuum Einsteinian spacetimes.

Though our proof is valid only for small data, the future completeness for large data is an open question which may have a positive answer.

7.1 Equations

A spacetime $(V_4, {}^{(4)}g)$ is said to possess a spacelike S^1, i.e. $U(1)$, isometry group if $V_4 = M \times R$, and M is an S^1 fiber bundle over a surface Σ (then V_4 is an S^1 fiber bundle over $V_3 = \Sigma \times R$), and ${}^{(4)}g$ is invariant under the action of the group S^1. Then, in a local trivialization of V_4 the metric reads:

$$ {}^{(4)}g = e^{-2\gamma}\,{}^{(3)}g + e^{2\gamma}(d\theta + A)^2, \tag{7.1} $$

where γ, ${}^{(3)}g$, A are respectively a scalar, a Lorentzian metric, and the representant of a S^1 connection 1 form, all on V_3.

Straightforward calculation shows that the vacuum Einstein equations on V_4, $Ricci({}^{(4)}g) = 0$ imply that the curvature $F := dA$ of the S^1 connection is such that:

$$ d(e^{4\gamma}\,{}^*F) = 0. \tag{7.2} $$

This equation is solved by setting

$$ F = e^{-4\gamma}\,{}^*d\omega \tag{7.3} $$

ω is a scalar function on V_3 called the twist potential.

The Einstein equations are then equivalent to:

1. The pair (γ, ω) satisfies a wave map equation from $(V_3, {}^{(3)}g)$ into the Poincaré plane (R^2, P), where:

$$P = 2(d\gamma)^2 + (1/2)e^{-4\gamma}(d\omega)^2. \tag{7.4}$$

2.. The Lorentzian metric ${}^{(3)}g$ satisfies the Einstein equations on V_3 with source the wave map:

$$Ricci({}^3g) - \rho = P(\partial_u \otimes \partial_u). \tag{7.5}$$

In 3 space dimensions the Ricci tensor is linearly equivalent to the Riemann tensor, one says sometimes that the Einstein equations in 2+1 dimensions are not dynamical. However the equivalence of the resolution of these 2+1 equations with the resolution of the constraints on each spacelike surface Σ_t depends on the topology of Σ. Indeed it can be shown that the resolution of the constraints on each surface Σ_t for the induced 2 - metric g and extrinsic curvature k gives that the projections on Σ_t, and the mixed projection on Σ_t and the normal to Σ_t, of the Einstein equations 5.8 are zero. The Bianchi identities show then that the projection on Σ_t of $Ricci({}^{(3)}g) - \rho$ is traceless, and also transverse, that is has a vanishing divergence in the metric g_t of Σ_t. In the case of a compact surface Σ, the space of such 2-tensor fields, called TT tensors, is finite dimensional, isomorphic to the space of classes of conformally equivalent metrics on Σ, called Teichmüller space, a manifold diffeomorphic to R^{6G-6}, if $G > 1$, G the genus of Σ. On the sphere ($G = 0$) there are no TT tensors, on the torus ($G = 1$) there are 2 linearly independent TT tensors. In the case $G > 0$ some equation must be added to the constraints to insure the verification of the full $2+1$ Einstein equations.

We consider the case where M is compact and $G > 1$. We find then that the 2+1 dimensional Einstein equations are equivalent to

a. Constraints on each Σ_t, which will be solved by the conformal method: one sets $g_t = e^{2\lambda}\sigma_t$, with σ_t some t dependent Riemannian metric on Σ. One finds that the constraints are equivalent to a semilinear elliptic system on each Σ_t, which determines g_t and k, modulo gauge conditions equivalent to elliptic linear equations, we choose in particular constant mean curvature for k, i.e. $tr_g k = \tau(t)$.

b. O.D. equations for the time evolution of the conformal geometry of (Σ_t, σ_t).

7.2 Local existence theorem.

The formulation given above, and known theorems for elliptic and hyperbolic equations, lead to the following theorem.

Theorem 7.1 *1. The Cauchy data on Σ_{t_0} are a C^∞ metric σ_0 and a TT tensor q_0, together with scalar functions, $(\gamma_0, \omega_0) \equiv u_0 \quad u(t_0, .) \in H_2, \dot{u}_0 \quad (N^{-1}e^{2\lambda}\partial_0 u)(t$ H_1. Then there exists a solution of the 2+1 Einstein -wave map equations on $\Sigma \times (t_0 \quad t_1, t_0 + t_2)$ taking these Cauchy data, if t_1 and t_2 are small enough.*

2. There exists a corresponding, vacuum, S^1 symmetric Einsteinian spacetime if the initial data satisfy an integrability condition for the construction of A.

7.3 Global existence

We have denoted by τ the mean extrinsic curvature of Σ_t. Hence Σ_t expands when τ increases from $\tau_0 < 0$ to zero. Set $t = -\tau^{-1}$, t increases from $t_0 > 0$ to infinity, Σ_t collapses when t tends to zero. We choose the future to be the expanding direction. The spacetime will manifest a singularity in the contracting direction, that is when $t = -\tau^{-1}$ tends to zero.

The future global existence is proved for small data by using the classical methods of energy estimates and bootstrap. A difficulty arises due to the fact that the metric σ_t, used in elliptic estimates, is itself an unknown which must be shown to be uniformly equivalent to its initial value. The proof of this equivalence requires the introduction of corrected energies.

Energies estimates.

One defines the first energy by using the Hamiltonian constraint of the 2+1 Einstein equations. One denotes by h the traceless part $k - \frac{1}{2}g\tau$ of k, and one sets:

$$\varepsilon^2 = E(t) = \int_{\Sigma_t}(|Du|^2 + |u'|^2 + \frac{1}{2}|h|^2)\mu_g$$

For the second energy, which has no obvious physical meaning, we take the usual second energy of a wave map, a hat denoting a gauge covariant derivative:

$$\varepsilon_1^2 = \tau^{-2}E^{(1)}(t) = \int_{\Sigma_t}(|\hat{\Delta}_g u|^2 + |\hat{D}u'|^2)\mu_g$$

Elliptic estimates deduced from the constraints and gauge equations give:

$$1 \leq \frac{1}{\sqrt{2}}|\tau|e^{\lambda} \leq 1 + C_{E,\sigma}(\varepsilon + \varepsilon_1),$$

$$0 \leq 2 - N \leq C_{E,\sigma}(\varepsilon^2 + \varepsilon\varepsilon_1),$$

where the $C_{E,\sigma}$ denote positive continuous functions of an a priori bound of $\varepsilon + \varepsilon_1$ and of the domain of σ in Teichmuller space.

Wave map estimates give then the equalities

$$\frac{d}{dt}E(t) \quad + \int_{\Sigma_t} N(|u'|^2 + \frac{1}{2}|h|^2)\mu_g \leq 0$$

$$\frac{dE^{(1)}}{dt} \quad 2\tau E^{(1)} \quad + \int_{\Sigma_t} N \mid Du' \mid^2 \mu_g + Z$$

where Z satisfy the inequality

$$|Z| \leq C_{\sigma,\Sigma}(\varepsilon + \varepsilon_1)^3.$$

The obtained bounds are not sufficient to show, using the ODE, that σ_t projects on a bounded subset of Teichmuller space.

Corrected energies.

To obtain bounds on the domain of σ, we need not only bounds on energies, but decay. This decay is proved by exploiting the negative (non definite) terms in the energies inequalities. One sets

$$E_\alpha(t) = E(t) - \alpha\tau \int_{\Sigma_t} (u - \bar{u})\, u' \mu_g$$

where $\bar{u}$ is the mean value of u, namely:

$$\bar{u} = \frac{1}{Vol_{\sigma_t}} \int_{\Sigma_t} u\mu_g, \quad u \equiv (\gamma, \omega)$$

and

$$E_\alpha^{(1)}(t) = E^{(1)}(t) + \alpha\tau \int_{\Sigma_t} \Delta_g u\, u' \mu_g$$

It can be shown that, for appropriately chosen α, the quantity $E_\alpha + \tau^{-2} E_\alpha^{(1)}$ bounds the total energy $\varepsilon^2 + \varepsilon_1^2$, and that elliptic estimates lead to inequalities for the corrected energies, which imply the following decay of the total energy:

$$(\varepsilon^2 + \varepsilon_1^2)(t) \le t^{-k} M (\varepsilon^2 + \varepsilon_1^2)(t_0), \quad k > 0.$$

This decay, in its turn, implies a bound of σ_t. The coefficient M in this inequality depends on the a priori bounds $C_{\Sigma,\sigma}$. A bootstrap argument shows the decay of the total energy if the initial total energy is small enough. The local existence theorem becomes then a future global existence theorem, for $t_0 \le t < +\infty$, for small initial data u_0 and $\dot{u}_0$. The obtained spacetime can be shown to be future complete.

Theorem 7.2 *There exists a future complete vacuum, S^1 symmetric Einsteinian spacetime corresponding to the Cauchy data σ_0, q_0, a smooth metric and TT tensor, and the functions (γ_0, ω_0): $u(t_0, .) \in H_2$, $\dot{u}_0 := (N^{-1} e^{2\lambda} \partial_0 u)(t_0, .) \in H_1$, if all these functions are small enough in norms.*

Progress about the global existence for large initial data of wave maps from a 2+1 dimensional manifold into a space of negative curvature will open the way to the removal on this smallness hypothesis.

2 Einstein equations.

A spacetime (V, g) is called Einsteinian if its metric satisfies the Einstein equations

$$Einst(g) + \Lambda g = T, \tag{2.1}$$

where $Einst(g)$ is the Einstein tensor, given in terms of the Ricci tensor $Ricci(g)$ and the scalar curvature $R(g)$ of the metric g by

$$Einst(g) := Ricci(g) - \frac{1}{2} g R(g). \tag{2.2}$$

Λ is a number called the cosmological constant. Its presence in the Einstein equations, and its possible values are under discussion. T is a symmetric 2-tensor, called the stress energy tensor, which models all the non gravitational energies, momenta and stresses. It is zero in vacuum.

8 Behaviour near the singularity.

A general conjecture is that the space interactions become negligible as one approaches the singularity: Einstein equations become equivalent to the ODE obtained in dropping space derivatives in these equations. A solution to these ODE is called a VTD (Velocity Terms Dominated) solution. Such a solution is called mixmaster, or BKL (Belinski, Kalatnikov, Lipshits) if it shows some kind of oscillatory behaviour when approaching the singularity. Work based on Hamiltonian methods (Damour and al.) supports the conjecture of a generic BKL behaviour for space times of dimension ≤ 10. The singularity is called AVTD (Asymptotic Velocity Terms Dominated) if the full Einstein equations admit a solution which tends asymptotically to a given VTD solution as one approaches the singularity. AVTD behaviour has been proved, by using Fuchsian type equations, for classes of 4 dimensional spacetimes with sources (Andersson-Rendall) or with $U(1)$ isometry group (CB, Isenberg, Moncrief).

9 Open problems.

Among the open problems is the rigorous mathematical study of the possible oscillating behaviour of the spacetimes near the big bang.

A problem for S^1 symmetric spacetimes with AVTD behaviour, is to match the solution deduced from a VTD given solution with the future complete solution.

The future evolution problem for arbitrary initial data and formation of singularities, together with the verification of the cosmic censorship conjectures, are still largely open, in spite of going on remarkable progresses made by excellent mathematicians using refined new techniques.

The Bianchi identities imply that the covariant divergence of the Einstein tensor is identically zero. The source tensor T has covariant divergence zero when the sources satisfy the relevant equations, Maxwell and Yang - Mills equations for electromagnetism and other fundamental interactions, or appropriate conservation equations in the case of macroscopic matter sources. These equations

$$\nabla . T = 0, \qquad (2.3)$$

make the system of equations 2.1, 2.3 coherent. They imply in particular that test particles follow geodesics of the metric, thus illustrating the local equivalence of inertial forces and gravitation.

When the source T is known, the Einstein equations are a system of quasilinear partial differential equations of the second order for the metric g. These equations are geometric equations, invariant by diffeomorphisms of V. Their resolution involves many problems linked with the theory of both hyperbolic and elliptic partial differential equations on manifolds.

It is still believed by the majority of physicists and astronomers that an Einsteinian manifold, eventually of dimensions greater than four, models the cosmos at all scales. Such manifolds model the primeval universe after the Planck time (about 10^{-43} seconds). Asymptotically Euclidean in space Einsteinian manifolds are used to study the motion of two isolated gravitating bodies. A representation agreeing with observations of the whole cosmos by an Einsteinian manifold is a subject of active debate, since, in spite of spectacular progress in astronomical observations, still little is known about our universe as a whole. It has become usual to call "cosmological" those Einsteinian spacetimes which have compact spacelike sections. However the cosmos we live in may well have non compact spacelike sections. It is legitimate for the mathematician to hope that his study of all possible models, with arbitrary topology will have some relevance to physics. This opens a vast field of investigations for the mathematician, where remarkable conjectures have been proposed. Many results have been obtained, sometimes surprising, but many fundamental questions remain open.

Among the most interesting questions debated today, are the questions of the beginning and the end of our universe. It is generally believed that it had a beginning. Most modern astronomers tell us the beginning was a big bang, that is, for the mathematician, a singularity of the spacetime. The ultimate future fate of the cosmos we live in cannot now really be predicted, in spite of the most recent observation data. For the mathematician the question is of the future global existence of a solution of the Einstein equations, starting from initial data. Since the cosmos is both a physical and geometrical object, the name "initial data" requires some thought, as well as the word "global", since the elapsed time depends on the trajectory of the observer. "Global" is usually taken as "completeness of causal geodesics".

3 Cauchy problem.

The Einstein equations are *geometric* equations, a proper formulation of the Cauchy problem for these equations is also geometric. On the other hand one expects solutions of Einstein equations, a classical, non quantum, field theory, to respect the relativistic causality principle. The corresponding spacetimes are globally hyperbolic, that is (Leray) the set of causal paths joining two points is compact in the Frechet topology. It has been proved by Geroch that globally hyperbolic spacetimes (M, g) are topological products $M \times R$, with each $M \times \{t\}$ intersected once by each inextendible timelike curve. Such submanifolds are called Cauchy surfaces.

An initial data set for the vacuum $(T = 0)$ Einstein equations is a triplet (M, g, K) where M is an n-dimensional manifold, g a properly Riemannian metric and K a symmetric 2-tensor on M. A development of an initial data set is an $n+1$ dimensional manifold V with a Lorentzian metric g such that M can be identified with a submanifold of V, g is the metric induced by g on M and K is identified with the *extrinsic curvature of M as submanifold of (V, g)*. A development is called Einsteinian if the metric g satisfies the Einstein equations.

Constraints.

The Cauchy data cannot be arbitrary, they must satisfy, on M, a scalar and a vector equation, called constraints; in vacuum

$$R(g) \quad |K|_g^2 + |Trace_g K|^2 \quad 0, \tag{3.1}$$

$$Div_g K \quad gradK \quad 0. \tag{3.2}$$

Modulo some arbitrary data, representing in a vague sense initial radiation data, the constraints can be put under the form of a semilinear elliptic system. Its resolution is fairly complete for compact or asymptotically Euclidean manifolds with small $Trace_g K$.

Evolution.

Due to diffeomorphisms invariance, the solution is not unique for the analyst. Its determination depends on a choice of gauge. When the local coordinates satisfy the wave equation in the spacetime metric g, the Einstein equations read as a quasidiagonal system of quasilinear wave equations for the components of g, quadratic in first derivatives of g :

$$g^{\alpha\beta}\partial^2_{\alpha\beta}g + Q^{\alpha\beta}(g)(\partial g, \partial g) = 0, \quad \partial_\alpha := \frac{\partial}{\partial x^\alpha} \tag{3.3}$$

The coordinate conditions and the constraints are preserved by such evolution, due to the Bianchi identities.

Theorem 3.1 *(local existence, global geometric uniqueness, vacuum case) An initial data set* $(g, K) \in H^{loc}_s \times H^{loc}_{s-1}$, $s > \frac{n}{2} + 1$ *satisfying the constraints admits a globally hyperbolic Einsteinian development, unique (up to isometries) in the class of maximal globally hyperbolic Einsteinian developments (CB and Geroch).*

4 Obstructions to future completeness.

The Hawking - Penrose singularity theorems, in the seventies, give various conditions under which the maximal future Einsteinian development of Cauchy data is incomplete for timelike or null geodesics, when the sources satisfy some positive energy hypothesis, because gravitation is then attractive and therefore has a tendency to focus the timelike, or null, geodesics (Raychaudhuri inequality, derived from the Jacobi identity satisfied by the second variation of the arc length in a Lorentzian manifold with Ricci tensor zero or satisfying some positivity condition). A sample of the hypotheses made in these theorems is

1. M has a constant positive mean extrinsic curvature.

or

2. The initial 3 - manifold M contains a closed trapped surface S, that is there exists an open relatively compact set in M whose boundary S is such that outgoing null geodesics at points of S are converging.

The problem with the application of the second singularity theorem to the global existence of a solution of the Cauchy problem is that it is not known how to control the formation of a trapped surface for generic initial data.