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Universality for Mathematical and Physical Systems

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(2) Gaussian orthogonal ensemble (GOE)—an ensemble of $N \times N$ real symmetric matrices $M = M^T = (M_{kj})$ with probability distribution

$$P_N(M)\,dM = \frac{1}{Z_N} e^{-\mathrm{tr} M^2}\, dM$$
$$= \frac{1}{Z_N} e^{-\mathrm{tr} M^2} \prod_{k \le j} dM_{kj}$$

$\mathrm{tr} M^2 \to \mathrm{tr} V(M)$, $V : \mathbb{R} \to \mathbb{R}$, $\lim_{|x| \to \infty} V(x) = +\infty \Rightarrow$ general Orthogonal Ensembles...

And of course, eig's $\lambda_1(M) \geq \cdots \geq \lambda_N(M)$ become random var's under GOE

We say that a system is modeled by RMT if it behaves statistically like the e'values of a "large" (GUE, GOE,...) random matrix.

There is something known as the standard procedure: suppose we wish to compare some statistical quantities $\{\alpha_k\}$ in the nbhood of some pt. A say, with the e'values $\{\lambda_k\}$ of some matrix in a nbhood of some energy E, say, then we always

center $\alpha_k \to \alpha_k - A$, $\lambda_k \to \lambda_k - E$ and

scale $\alpha_k \to \tilde{\alpha}_k = \gamma_A(\alpha_k - A)$, $\lambda_k \to \tilde{\lambda}_k = \gamma_\lambda(\lambda_k - E)$

so that

$$\mathbb{E}\{\#\text{ of } \tilde{\alpha}_k\text{'s/unit interval}\} = \mathbb{E}\{\#\text{ of } \tilde{\lambda}_k\text{'s/unit interval}\} = 1$$

So <u>whenever</u> we compare some phys./math. system with an eigenvalue ensemble, we'll always follow the standard procedure.

We are interested in 2 particular statistics for GUE (there are similar formulae for GOE, but we won't write them down). Let $\theta > 0$, and define the <u>gap probability</u>

$$P_N(\theta) = \mathbb{P}\{M \in \text{GUE} : M \text{ has no eig's in } (-\theta, \theta)\}.$$

Let γ_N be the appropriate scaling for the standard procedure.

Gaudin and Mehta showed that for $y > 0$,

$$\lim_{N\to\infty} P_N(y/\gamma_N) = \det(1 - K_y)$$

where K_y denotes the trace-class operator with kernel

$$K_y(u, v) = \frac{\sin \pi(u - v)}{\pi(u - v)}$$

acting on $L^2(-y, y)$.

Regarding the eigenvalue $\lambda_1(M)$ at the top of the spectrum of M, we have the following result of Tracy and Widom:

$$\lim_{N\to\infty} \mathbb{P}\left\{ M \in \text{GUE} : \frac{\lambda_1 - \sqrt{2N}}{2^{-1/2}N^{-1/6}} \le t \right\}$$
$$= F(t) \qquad (= \text{Tracy-Widom distribution})$$
$$= e^{-\int_t^\infty (s-t)u(s)^2\,ds},$$

where $u(s)$ is the (unique, global) solution of the Painlevé II eqn.:

$$u''(s) = 2u(s)^3 + su(s), \quad u(s) \sim Ai(s) \text{ as } s \to +\infty.$$

In addition, we note the characteristic feature of GUE, GOE,... viz. **repulsion**:

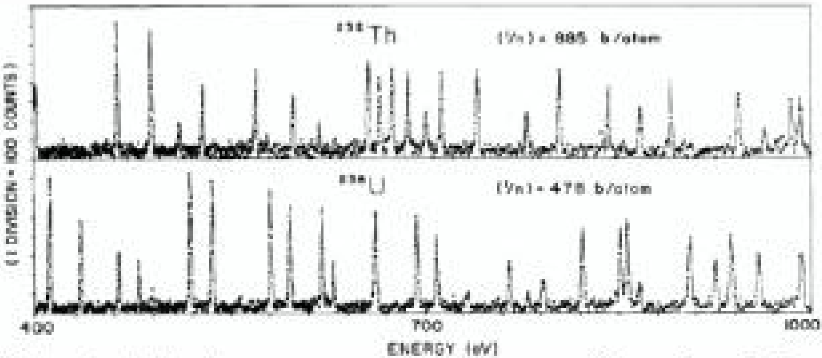
$$P(\lambda_1, ..., \lambda_N)\, d\lambda = c \prod_{i<j} |\lambda_i - \lambda_j|^\beta \, d\lambda$$

$\beta = 1$	GOE
$\beta = 2$	GUE
($\beta = 4$	GSE)

Thus the probability that two eigenvalues are close is **small**!

C. Examples of mathematical/physical systems

(1) Consider scattering of neutrons ν off heavy nuclei, such as Th^{232} or U^{238}.



Rahn et al, Neutron resonance spectroscopy, *Phys. Rev. C* 6 (1972), p. 1857

Question: How does one model the resonance peaks?

(2) Here we consider the work of H. Montgomery in the early 1970's on the zeros of the Riemann zeta function $\zeta(s)$. Assuming the Riemann hypothesis, Montgomery rescaled the imaginary parts $\gamma_1 \leq \gamma_2 \leq \cdots$ of the (nontrivial) zeros $\{\frac{1}{2} + i\gamma_j\}$ of $\zeta(s)$,

$$\gamma_j \to \tilde{\gamma}_j = \frac{\gamma_j \log \gamma_j}{2\pi}$$

to have mean spacing 1 as $T \to \infty$, i.e.

$$\lim_{T\to\infty} \frac{\#\{j \geq 1 : \tilde{\gamma}_j \leq T\}}{T} = 1.$$

For any $a < b$, he then computed the two-point correlation function for the $\tilde{\gamma}_j$'s

$$\#\{\text{ordered pairs}\,(j_1, j_2), j_1 \neq j_2 : 1 \leq j_1, j_2 \leq N, \tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_2} \in (a, b)\}$$

He showed, modulo certain technical restrictions, that

$$R(a,b) \equiv \lim_{N\to\infty} \frac{1}{N} \#\{\text{ordered pairs}\,(j_1, j_2), j_1 \neq j_2 :$$
$$1 \le j_1, j_2 \le N, \tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_2} \in (a,b)\}$$

exists and is given by a certain explicit formula.

<u>Question:</u> What formula did Montgomery obtain for $R(a,b)$?

(3) Card game–patience sorting

Consider a deck of N cards numbered for convenience $1, 2, \ldots, N$. Shuffle the deck and play the following game: place the top card... $\Rightarrow$ # of piles $\equiv q_N = q_N(\pi)$, $\pi \in S_N$.

Example: consider $N = 6$. After shuffle $\pi \in S_N$ suppose we get $3, 4, 1, 5, 6, 2$. Then, sorting

		1	1	1	1 2
3	3 4	3 4	3 4 5	3 4 5 6	3 4 5 6

and so $q_6(\pi) = 4$ piles.

Equip S_N with uniform measure.
Question: How does $q_N(\pi)$ vary statistically as $N \to \infty$?

(4) Buses in Cuernavaca, Mexico

There is a town in Mexico called Cuernavaca (pop. ~500,000). No municipal bus system, but individual operators: Poisson phenomena, bunching and long waits. What to do?

Hire observers: tell drivers to ↑ or ↓ ⇒ steady and reliable bus service! Citizens of Cuernevaca pretty pleased with their bus service. Recently, 2 Czech physicists, Krbálek and Šeba, went to Mexico to investigate. Took data for ~1 month, on Route 4.

Question: What did they find?

Outline of talk:

A. General discussion of universality

B. A mathematical model: random matrix theory (RMT)

C. Some physical and mathematical systems

D. Solution of **C.**

E. Mathematical methods

(5) Walker model (M. Fisher)

Suppose we have walkers on $\mathbb{Z}$ located initially at $0, 1, 2, \ldots$, subject to the following rules:
(a) at each integer time k one walker makes a step to the left;
(b) no 2 walkers can occupy the same site ("vicious walkers");
(c) the walker that moves at time k is chosen "randomly".

Example: consider the following walk from time $t = 0$ to time $t = 4$:

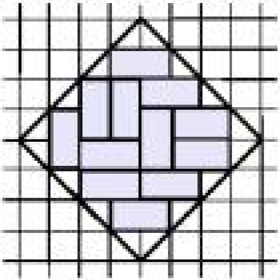
$\cdot$	$\cdot$	$\times$	$\cdot$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$t = 4$
$\cdot$	$\cdot$	$\times$	$\cdot$	$\times$	$\cdot$	$\times$	$\times$	$\times$	$t = 3$
$\cdot$	$\cdot$	$\cdot$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$\times$	$t = 2$
$\cdot$	$\cdot$	$\cdot$	$\times$	$\cdot$	$\times$	$\times$	$\times$	$\times$	$t = 1$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\times$	$\times$	$\times$	$\times$	$t = 0$
-4	-3	-2	-1	0	1	2	3	4	

Let d_N be the distance moved by the the 0-walker at time $t = N$. (Here $d_4 = 2$.)

Question: How does d_N behave statistically as $N \to \infty$?

(6) Aztec diamond

Consider tilings of the (tilted) square of size $n+1$ by horizontal and vertical dominoes, e.g. for $n+1=4$:



- dominos contained in the square;
- $2^{n(n+1)/2}$ tilings;
- assume all tilings are equally likely.

Question: What does a typical tiling look like as $n \to \infty$?

(7) Airline boarding (E. Bachmat et al.)

A problem familiar to all of us–how long does it take to board an airplane?

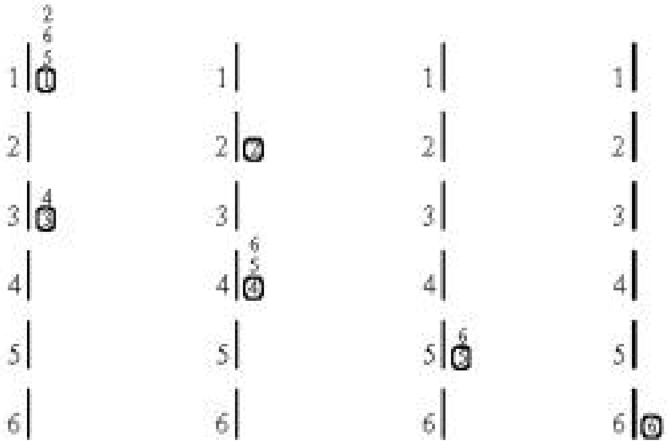
Simplifying assumptions (can make model far more realistic...):

– 1 seat/row

– passengers are very thin

– passengers move very quickly. Bulk of time -1 unit- is taken to store luggage.

Example: $N = 6$ passengers; order at the gate (3 4 1 5 6 2).



D. Solutions of Problems (C)

The remarkable fact is that all these systems are modeled statistically by random matrix theory. More precisely:

(1) scattering resonances: after the standard procedure, the probability that there are no resonances in the interval $(-y, y)$ is given either by $\det(1 - K_y)$, which is the (asymptotic) gap probability for GUE introduced above, or by its GOE analog, depending on certain symmetry conditions.

(2) <u>zeta function:</u> the formula that Montgomery obtained (modulo some technicalities) for the limiting 2-point correlation function $R(a,b)$ for the zeros of the zeta function was

$$R(a,b) = \int_a^b 1 - \left(\frac{\sin(\pi r)}{\pi r}\right)^2 dr.$$

As noted by Dyson, this is precisely the limiting 2-point correlation function for the eigenvalues of a random GUE matrix (later Rudnick, Sarnak, Katz, Keating,...)

(3) <u>patience sorting:</u> $P_N(\pi) = \#$ of piles behaves like the largest GUE eigenvalue $\lambda_1(M)$, i.e

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{P_N - 2\sqrt{N}}{N^{1/6}} \le t\right) = F(t) = \text{TW distribution}$$

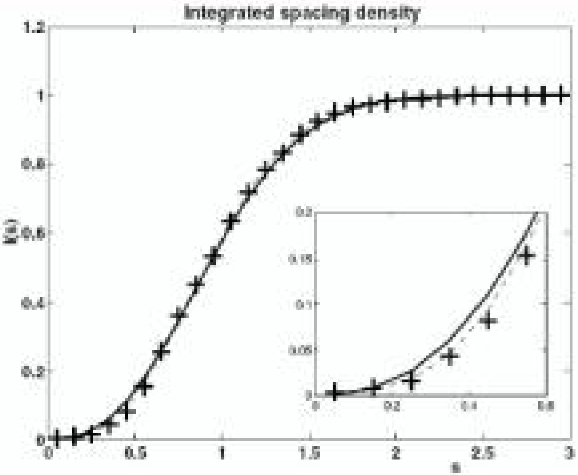
(4) buses in Cuernavaca: Krbálek and Šeba found that the spacing distribution between buses behaves statistically like the spacings between the eigenvalues of a random GUE matrix given by $\int_0^s p(u)\,du$, where

$$p(u) = \frac{d^2}{du^2}\,\mathrm{det}\Big(1 - K \restriction L^2(0,u)\Big)$$

and

$$K(x,y) = \frac{\sin\pi(x-y}{\pi(x-y)}.$$

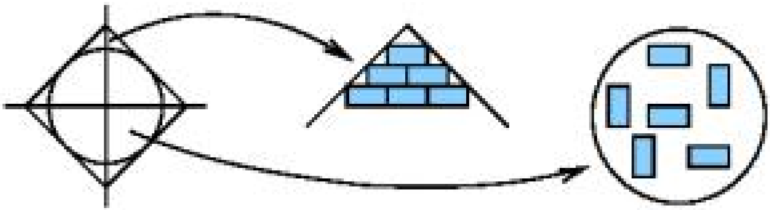
The data:



M. Krbálek and P. Šeba, The statistical properties of the city transport in Cuernavaca (Mexico) and random matrix ensembles, *J. Phys. A: Math. Gen.* **33** (2000), 229–234.

(5) walkers: d_N behaves - see Forrester - like the largest eig. of GOE
i.e., after centering and scaling d_N appropriately, it converges in distribution to $F_1(t)$, the TW-distribution for the largest GOE eigenvalue (similar to $F(t)$; also involves Painlevé II solution).

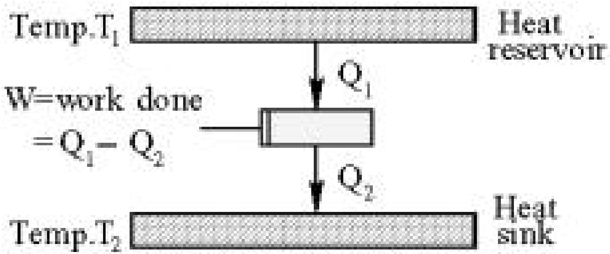
(6) aztec diamond: rescaling as $n \to \infty$, $x \to x/n + 1$, an arctic circle emerges (Elkies, Propp et.al.)



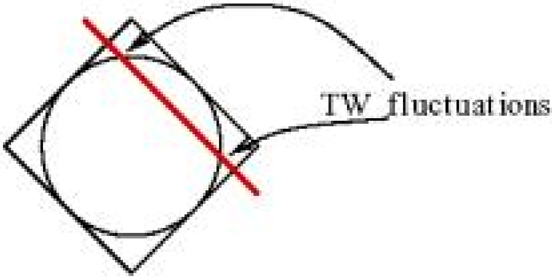
A. General discussion

All physical systems in equilibrium obey the laws of thermodynamics (TD^Y):
Most familiar – conservation of energy
Another law – so-called 2nd law of TD^Y; tells us the following:

Suppose we have a heat engine:



Inside polar regions, tiling is frozen, while inside temperate zone it is chaotic.
Johansson has proved that fluctuations of the arctic circle follow Tracy-Widom.



(7) airline boarding: Again, after centering and scaling appropriately, T_N, the boarding time, converges in distribution to $F(t)$, the Tracy-Widom distribution.

E. Mathematical methods

Many other problems: hexagonal tilings, condensation, percolation, L functions,...

Status of the problems:

(1) neutron scattering – experimental/numerical

(2) zeta function – theorem (modulo restrictions)

(3) patience sorting – theorem

(4) buses in Cuernavaca – model + theorem

(5) walkers – theorem

(6) aztec diamond – theorem

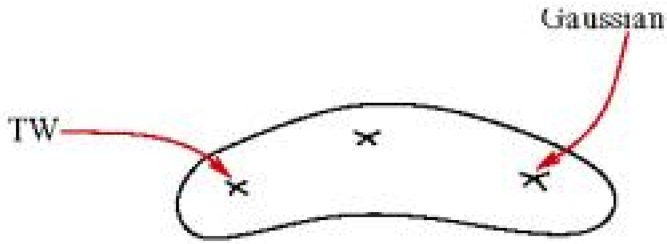
(7) airline boarding – theorem

What kind of mathematics is involved here?

Integrable systems is the key player:
–inverse scattering theory
–Riemann-Hilbert methods
–Painlevé theory, determinants
–classical and Riemann-Hilbert steepest descent method
–combinatorial identities: Gessel,...

Intuitions: intrinsic probabilistic point of view:
Baik-Suidan, Bodineau-Martin

Space of probability distributions: natural arena:



"Macroscopic" mathematics?
–some systems/examples isomorphic, others not.

Efficiency % of the conversion of heat into work is

$$\% = W/Q_1.$$

2^{nd} law: if very careful, no loss to friction, then the **maximal** efficiency is given by

$$(\%)_{\text{max}} = \frac{T_1 - T_2}{T_1}$$

Nature is set up so we can't do any better.

On the other hand, it is a very old thought ($\leq$ Democritus) that matter, hard matter like this table, is built out of tiny constituents–atoms–obeying their own laws of interaction.

The juxtaposition of these 2 points of view
macroscopic world of tangible objects $\longleftrightarrow$ microscopic world of atoms

$\Rightarrow$ fundamental, continuing challenge to scientists/philosophers; in particular

How does one derive the **macro.** laws of TD^Y from the **micro.** laws of atoms?

Special, salient feature of this challenge is that
the **same** laws of TD^Y should emerge no matter what the details of the atomic interaction.

In the world of physics this is known as

UNIVERSALITY

<u>Caveat:</u> physicists usually use "universality" in more restricted sense, critical phenomena with the same scaling laws..., but...

<u>Sub-universality classes:</u> water, vinegar,... obey Navier-Stokes eqns.; heavy oils obey lubrication eqns.

Until recently, this kind of thinking not common in the world of math. Mathematicians think of their problems as sui generis, with special features... Two problems are the "same" only if one can establish some isomorphism between them... But in the last few years, universality has been emerging in the problems they consider: the goal is to illustrate some of these developments.

Mathematical precendents: central limit theorem↔de Moivre-Laplace: $\{x_i\}$ i.i.d., mean 0, variance 1,

$$\mathbb{P}\left(\frac{x_1 + \cdots x_n}{\sqrt{n}} \leq t\right) \to N(0,1)$$

B. A mathematical model: RMT

At this point, many different random matrix models are of interest. We will be interested here primarily in 2 ensembles

(1) Gaussian Unitary Ensemble (GUE)–an ensemble of $N \times N$ Hermitian matrices $M = M^* = \left(M_{kj}\right)$ with probability distribution

$$
\begin{aligned}
P_N(M)\,dM &= \frac{1}{Z_N} e^{-\mathrm{tr} M^2}\,dM \\
&= \frac{1}{Z_N} e^{-\mathrm{tr} M^2} \prod_{k=1}^{N} dM_{kk} \times \prod_{k<j} d\,\mathrm{Re}\, M_{jk} \\
&\qquad \times \prod_{k<j} d\,\mathrm{Im}\, M_{jk}
\end{aligned}
$$

If replace $\mathrm{tr}M^2 \to \mathrm{tr}V(M)$, $V : \mathbb{R} \to \mathbb{R}$,
$\lim_{|x|\to\infty} V(x) = +\infty$, get general Unitary Ensemble:
"unitary" refers to invariance under unitary conjugation.

Eigenvalues: $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_N(M)$

Under GUE, $\{\lambda_i(M)\}$ become random variables