# Kähler manifolds and transcendental techniques in algebraic geometry

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# Sheaf / De Rham / Dolbeault / cohomology

- Sheaf cohomology $H^q(X, \mathcal{F})$
  especially when $\mathcal{F}$ is a coherent analytic sheaf.
- Special case : cohomology groups $H^q(X, R)$ with values in constant coefficient sheaves $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$
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- $\Omega^p_X = \mathcal{O}(\Lambda^p T^*_X) =$ sheaf of holomorphic $p$-forms on $X$.
- Cohomology classes [forms / currents yield same groups]

$$\alpha \ d\text{-closed } k\text{-form/current to } \mathbb{C} \longmapsto \{\alpha\} \in H^{p+q}(X, \mathbb{C})$$

$$\alpha \ \overline{\partial}\text{-closed } (p,q)\text{-form/current to } F \longmapsto \{\alpha\} \in H^{p,q}(X, F)$$

Dolbeault isomorphism (Dolbeault - Grothendieck 1953)

$$H^{0,q}(X, F) \simeq H^{0,q}(X, \mathcal{O}(F)),$$

$$H^{p,q}(X, F) \simeq H^{0,q}(X, \Omega^p_X \otimes \mathcal{O}(F))$$

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# Hodge decomposition theorem

- **Theorem.** *If $(X, \omega)$ is compact Kähler, then*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}).$$

- *Each group $H^{p,q}(X, \mathbb{C})$ is isomorphic to the space of $(p, q)$ harmonic forms $\alpha$ with respect to $\omega$, i.e. $\Delta_\omega \alpha = 0$.*
- **Hodge Conjecture** [a millenium problem!].
  *If $X$ is a projective algebraic manifold, Hodge $(p, p)$-classes $= H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Q})$ are generated by classes of algebraic cycles of codimension $p$ with $\mathbb{Q}$-coefficients.*
- **(Claire Voisin, 2001)** $\exists$ *4-dimensional complex torus $X$ possessing a non trivial Hodge class of type $(2, 2)$, such that every coherent analytic sheaf $\mathcal{F}$ on $X$ satisfies $c_2(\mathcal{F}) = 0$.*

# Idea of proof of Claire Voisin's counterexample

The idea is to show the existence of a 4-dimensional complex torus $X = \mathbb{C}^4/\Lambda$ which does not contain any analytic subset of positive dimension, and such that the Hodge classes of degree 4 are perpendicular to $\omega^{n-2}$ for a suitable choice of the Kähler metric $\omega$.

The lattice $\Lambda$ is explicitly found via a number theoretic construction of Weil based on the number field $\mathbb{Q}[i]$, also considered by S. Zucker.

The theorem of existence of Hermitian Yang-Mills connections for stable bundles combined with Lübke's inequality then implies $c_2(\mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on the torus.

# Kodaira embedding theorem

**Theorem.** *$X$ a compact complex $n$-dimensional manifold. Then the following properties are equivalent.*

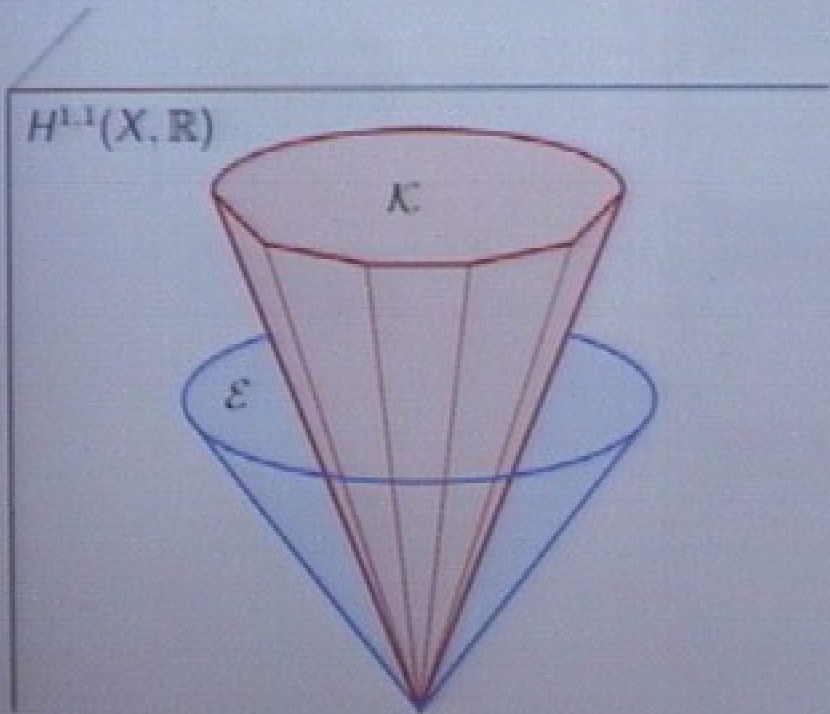
- *$X$ can be embedded in some projective space $\mathbb{P}^N_{\mathbb{C}}$ as a closed analytic submanifold (and such a submanifold is automatically algebraic by Chow's thorem).*
- *$X$ carries a hermitian holomorphic line bundle $(L, h)$ with positive definite smooth curvature form $i\Theta_{L,h} > 0$.*
  For $\xi \in L_x \simeq \mathbb{C}$, $\|\xi\|_h^2 = |\xi|^2 e^{-\varphi(x)}$,
  $$i\Theta_{L,h} = i\partial\overline{\partial}\varphi = -i\partial\overline{\partial}\log h,$$
  $$c_1(L) = \left\{\frac{i}{2\pi}\Theta_{L,h}\right\}.$$
- *$X$ possesses a Hodge metric, i.e., a Kähler metric $\omega$ such that $\{\omega\} \in H^2(X, \mathbb{Z})$.*

# Positive cones

**Definition.** Let $X$ be a compact Kähler manifold.

- The Kähler cone is the set $\mathcal{K} \subset H^{1,1}(X,\mathbb{R})$ of cohomology classes $\{\omega\}$ of Kähler forms. This is an open convex cone.
- The pseudo-effective cone is the set $\mathcal{E} \subset H^{1,1}(X,\mathbb{R})$ of cohomology classes $\{T\}$ of closed positive $(1,1)$ currents. This is a closed convex cone.
  (by weak compactness of bounded sets of currents).
- Always true: $\overline{\mathcal{K}} \subset \mathcal{E}$.
- One can have: $\overline{\mathcal{K}} \subsetneq \mathcal{E}$:
  if $X$ is the surface obtained by blowing-up $\mathbb{P}^2$ in one point, then the exceptional divisor $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

# Kähler (red) cone and pseudoeffective (blue) cone



# Neron Severi parts of the cones

In case $X$ is projective, it is interesting to consider the "algebraic part" of our "transcendental cones" $\mathcal{K}$ and $\mathcal{E}$, which consist of suitable integral divisor classes. Since the cohomology classes of such divisors live in $H^2(X,\mathbb{Z})$, we are led to introduce the Neron-Severi lattice and the associated Neron-Severi space

$$
\begin{aligned}
\mathrm{NS}(X) &:= H^{1,1}(X,\mathbb{R}) \cap \left(H^2(X,\mathbb{Z})/\{\text{torsion}\}\right),\\
\mathrm{NS}_{\mathbb{R}}(X) &:= \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R},\\
\mathcal{K}_{\mathrm{NS}} &:= \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X),\\
\mathcal{E}_{\mathrm{NS}} &:= \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X).
\end{aligned}
$$

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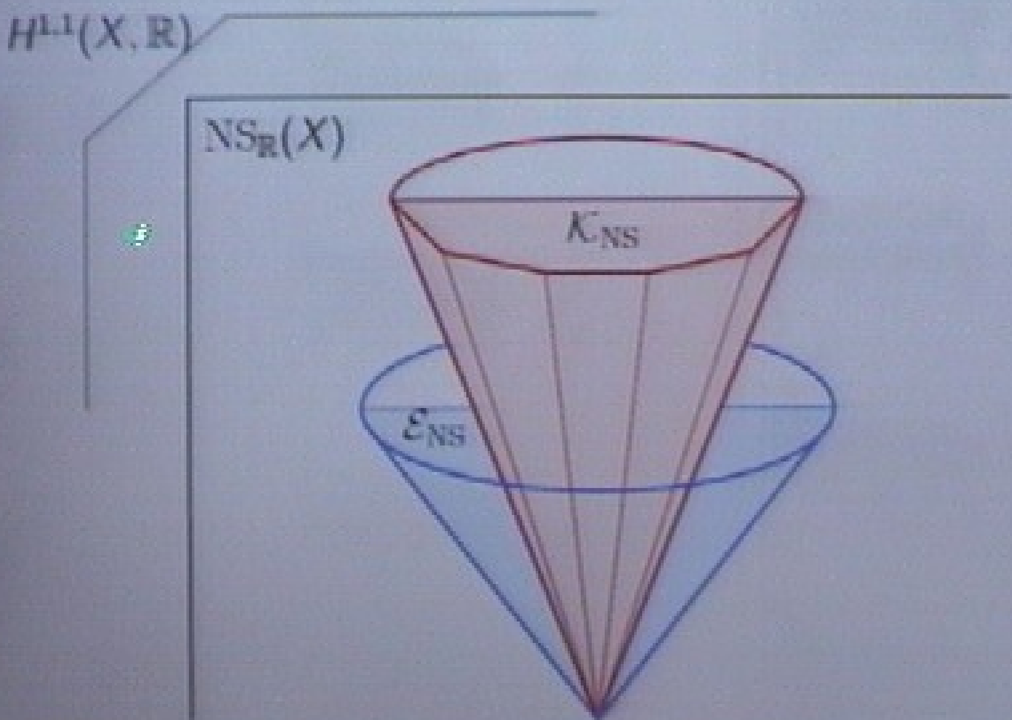

# Complex manifolds / $(p, q)$-forms

- Goal : study the geometric / topological / cohomological properties of compact Kähler manifolds
- A complex $n$-dimensional manifold is given by coordinate charts equipped with
  local holomorphic coordinates $(z_1, z_2, \ldots, z_n)$.
- A differential form $u$ of type $(p, q)$ can be written as a sum

$$u(z) = \sum_{|J|=p,|K|=q} u_{JK}(z)\, dz_J \wedge d\overline{z}_K$$

where $J = (j_1, \ldots, j_p)$, $K = (k_1, \ldots, k_q)$,

$$dz_J = dz_{j_1} \wedge \ldots \wedge dz_{j_p}, \quad d\overline{z}_K = d\overline{z}_{k_1} \wedge \ldots \wedge d\overline{z}_{k_q}.$$

# ample / nef / effective / big divisors

**Theorem** (Kodaira+successors, D90). *Assume $X$ projective.*

- *$\mathcal{K}_{\mathrm{NS}}$ is the open cone generated by* ample *(or* very ample*) divisors $A$ (Recall that a divisor $A$ is said to be very ample if the linear system $H^0(X, \mathcal{O}(A))$ provides an embedding of $X$ in projective space).*
- *The closed cone $\overline{\mathcal{K}}_{\mathrm{NS}}$ consists of the closure of the cone of* nef divisors *$D$ (or nef line bundles $L$), namely effective integral divisors $D$ such that $D \cdot C \geq 0$ for every curve $C$.*
- *$\mathcal{E}_{\mathrm{NS}}$ is the closure of the cone of* effective divisors*, i.e. divisors $D = \sum c_j D_j$, $c_j \in \mathbb{R}_+$.*
- *The interior $\mathcal{E}^{\circ}_{\mathrm{NS}}$ is the cone of* big divisors*, namely divisors $D$ such that $h^0(X, \mathcal{O}(kD)) \geq c\, k^{\dim X}$ for $k$ large.*

Proof: $L^2$ estimates for $\overline{\partial}$ / Bochner-Kodaira technique

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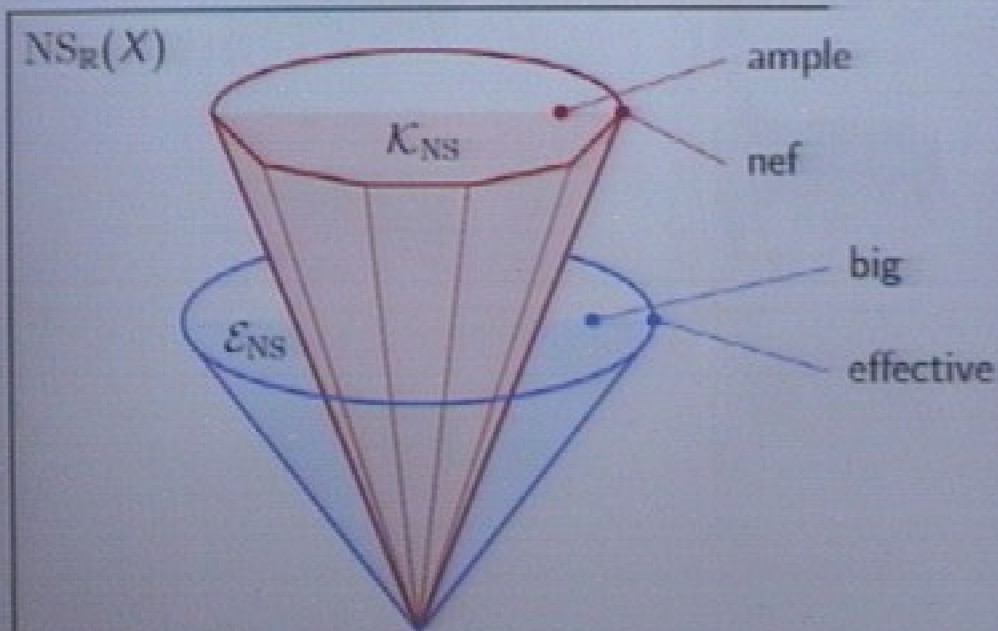

# Approximation of currents, Zariski decomposition

- **Definition.** *On $X$ compact Kähler, a* ***Kähler current*** *$T$ is a closed positive $(1,1)$-current $T$ such that $T \geq \delta\omega$ for some smooth hermitian metric $\omega$ and a constant $\delta \ll 1$.*

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- **Theorem.** $\alpha \in \mathcal{E}^\circ \Leftrightarrow \alpha = \{T\}$, $T =$ *a Kähler current.*

  We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.

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  We say that $\mathcal{E}^\circ$ is the cone of big (1,1)-classes.
- **Theorem** (D92). *Any Kähler current $T$ can be written*

$$T = \lim T_m$$

*where $T_m \in \alpha = \{T\}$ has logarithmic poles, i.e. $\exists$ a modification $\mu_m : \widetilde{X}_m \to X$ such that*

$$\mu_m^* T_m = [E_m] + \beta_m$$

*where $E_m$ is an effective $\mathbb{Q}$-divisor on $\widetilde{X}_m$ with coefficients in $\frac{1}{m}\mathbb{Z}$ and $\beta_m$ is a Kähler form on $\widetilde{X}_m$.*

# Idea of proof of analytic Zariski decomposition (1)

Locally one can write $T = i\partial\overline{\partial}\varphi$ for some strictly plurisubharmonic potential $\varphi$ on $X$. The approximating potentials $\varphi_m$ of $\varphi$ are defined as

$$\varphi_m(z) = \frac{1}{2m}\log\sum_{\ell}|g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \{f \in \mathcal{O}(\Omega) ; \int_{\Omega}|f|^2 e^{-2m\varphi}dV < +\infty\}.$$

The Ohsawa-Takegoshi $L^2$ extension theorem (applied to extension from a single isolated point) implies that there are enough such holomorphic functions, and thus $\varphi_m \geq \varphi - C/m$. On the other hand $\varphi = \lim_{m\to+\infty}\varphi_m$ by a Bergman kernel trick and by the mean value inequality.

# Idea of proof of analytic Zariski decomposition (2)

The Hilbert basis $(g_{\ell,m})$ is a family of local generators of the multiplier ideal sheaf $\mathcal{I}(mT) = \mathcal{I}(m\varphi)$. The modification $\mu_m : \widetilde{X}_m \to X$ is obtained by blowing-up this ideal sheaf, with

$$\mu_m^* \mathcal{I}(mT) = \mathcal{O}(-mE_m).$$

for some effective $\mathbb{Q}$-divisor $E_m$ with normal crossings on $\widetilde{X}_m$. Now, we set $T_m = i\partial\overline{\partial}\varphi_m$ and $\beta_m = \mu_m^* T_m - [E_m]$. Then $\beta_m = i\partial\overline{\partial}\psi_m$ where

$$\psi_m = \frac{1}{2m}\log\sum_\ell |g_{\ell,m}\circ\mu_m/h|^2 \quad \text{locally on } \widetilde{X}_m$$

and $h$ is a generator of $\mathcal{O}(-mE_m)$, and we see that $\beta_m$ is a smooth semi-positive form on $\widetilde{X}_m$. The construction can be made global by using a gluing technique, e.g. via partitions of unity, and $\beta_m$ can be made Kähler by a perturbation argument.

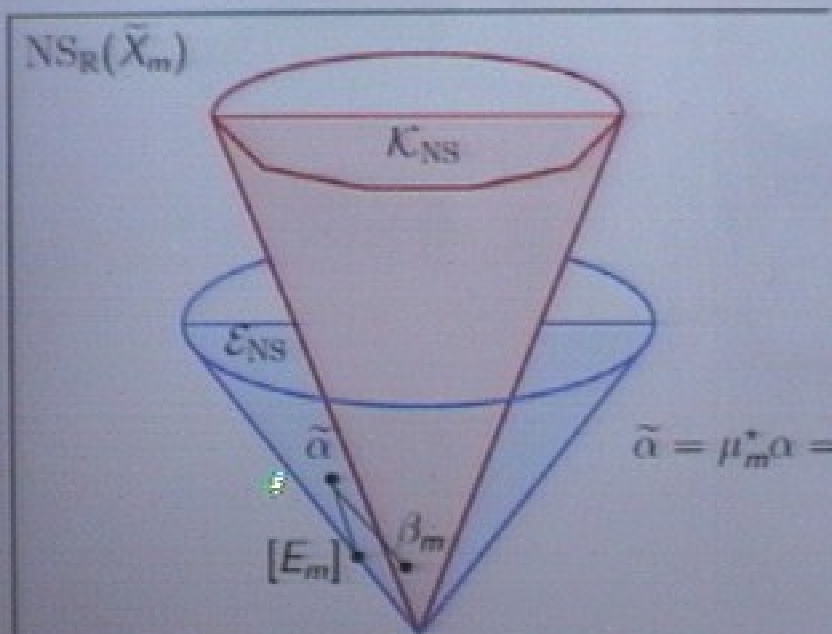
# Algebraic analogue

The more familiar algebraic analogue would be to take $\alpha = c_1(L)$ with a big line bundle $L$ and to blow-up the base locus of $|mL|$, $m \gg 1$, to get a $\mathbb{Q}$-divisor decomposition

$$\mu_m^* L \sim E_m + D_m, \qquad E_m \text{ effective}, \quad D_m \text{ free}.$$

Such a blow-up is usually referred to as a "log resolution" of the linear system $|mL|$, and we say that $E_m + D_m$ is an approximate Zariski decomposition of $L$. We will also use this terminology for Kähler currents with logarithmic poles.

# Analytic Zariski decomposition




$\widetilde{\alpha} = \mu_m^* \alpha = [E_m] + \beta_m$

# Characterization of the Fujiki class $\mathcal{C}$

**Theorem** (Demailly-Păun 2004). *A compact complex manifold $X$ is bimeromorphic to a Kähler manifold $\widetilde{X}$ (or equivalently, dominated by a Kähler manifold $\widetilde{X}$) if and only if it carries a Kähler current $T$.*

*Proof.* If $\mu : \widetilde{X} \to X$ is a modification and $\widetilde{\omega}$ is a Kähler metric on $\widetilde{X}$, then $T = \mu_*\widetilde{\omega}$ is a Kähler current on $X$.

Conversely, if $T$ is a Kähler current, we take $\widetilde{X} = \widetilde{X}_m$ and $\widetilde{\omega} = \beta_m$ for $m$ large enough.

**Definition.** *The class of compact complex manifolds $X$ bimeromorphic to some Kähler manifold $\widetilde{X}$ is called the Fujiki class $\mathcal{C}$.*
*Hodge decomposition still holds true in $\mathcal{C}$.*

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# Numerical characterization of the Kähler cone

**Theorem** (Demailly-Păun 2004).
*Let X be a compact Kähler manifold. Let*

$$\mathcal{P} = \left\{\alpha \in H^{1,1}(X,\mathbb{R}) ;\ \int_Y \alpha^p > 0,\ \forall Y \subset X,\ \dim Y = p\right\}.$$

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**Corollary** (Projective case).
*If X is projective algebraic, then $\mathcal{K} = \mathcal{P}$.*

*Note:* this is a "transcendental version" of the Nakai-Moishezon criterion.

# Example (non projective) for which $\mathcal{K} \subsetneq \mathcal{P}$.

Take $X$ = generic complex torus $X = \mathbb{C}^n/\Lambda$.

Then $X$ does not possess any analytic subset except finite subsets and $X$ itself.

Hence $\mathcal{P} = \{\alpha \in H^{1,1}(X,\mathbb{R}) : \int_X \alpha^n > 0\}$.

Since $H^{1,1}(X,\mathbb{R})$ is in one-to-one correspondence with constant hermitian forms, $\mathcal{P}$ is the set of hermitian forms on $\mathbb{C}^n$ such that $\det(\alpha) > 0$, i.e.
possessing an even number of negative eigenvalues.

$\mathcal{K}$ is the component with all eigenvalues $> 0$.

# Proof of the theorem : use Monge-Ampère

Fix $\alpha \in \overline{\mathcal{K}}$ so that $\int_X \alpha^n > 0$.

If $\omega$ is Kähler, then $\{\alpha + \varepsilon\omega\}$ is a Kähler class $\forall \varepsilon > 0$.

Use the Calabi-Yau theorem (Yau 1978) to solve the Monge-Ampère equation

$$(\alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon)^n = f_\varepsilon$$

where $f_\varepsilon > 0$ is a suitably chosen volume form.

Necessary and sufficient condition :

$$\int_X f_\varepsilon = (\alpha + \varepsilon\omega)^n \quad \text{in } H^{n,n}(X, \mathbb{R}).$$

Otherwise, the volume form of the Kähler metric $\alpha_\varepsilon = \alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon$ can be spread randomly.

## Proof of the theorem : concentration of mass

In particular, the mass of the right hand side $f_\varepsilon$ can be spread in an $\varepsilon$-neighborhood $U_\varepsilon$ of any given subvariety $Y \subset X$.

If $\operatorname{codim} Y = p$, on can show that

$$(\alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon)^p \to \Theta \quad \text{weakly}$$

where $\Theta$ positive $(p, p)$-current and $\Theta \geq \delta[Y]$ for some $\delta > 0$.

Now, "diagonal trick": apply the above result to

$$\widetilde{X} = X \times X, \qquad \widetilde{Y} = \text{diagonal} \subset \widetilde{X}, \qquad \widetilde{\alpha} = \mathrm{pr}_1^*\alpha + \mathrm{pr}_2^*\alpha.$$

As $\widetilde{\alpha}$ is nef on $\widetilde{X}$ and $\int_{\widetilde{X}}(\widetilde{\alpha})^{2n} > 0$, it follows by the above that the class $\{\widetilde{\alpha}\}^n$ contains a Kähler current $\Theta$ such that $\Theta \geq \delta[\widetilde{Y}]$ for some $\delta > 0$. Therefore

$$T := (\mathrm{pr}_1)_*(\Theta \wedge \mathrm{pr}_2^*\omega)$$

is numerically equivalent to a multiple of $\alpha$ and dominates $\delta\omega$, and we see that $T$ is a Kähler current.

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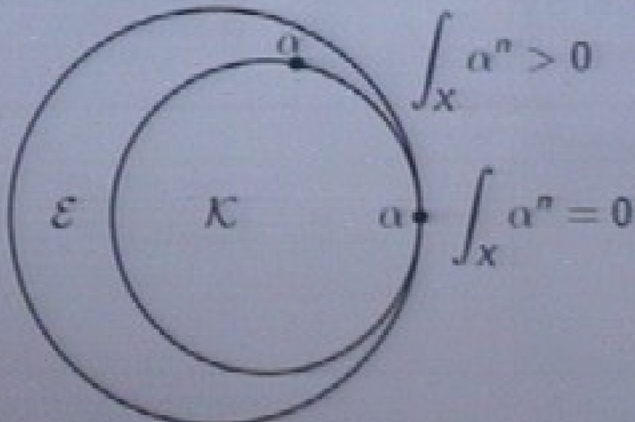
# Generalized Grauert-Riemenschneider result

**Main conclusion** (Demailly-Păun 2004).
*Let $X$ be a compact Kähler manifold and let $\{\alpha\} \in \overline{\mathcal{K}}$ such that $\int_X \alpha^n > 0$.*
*Then $\{\alpha\}$ contains a Kähler current $T$, i.e. $\{\alpha\} \in \mathcal{E}^\circ$.*



# Complex manifolds / Currents

- A current is a differential form with distribution coefficients

$$T(z) = \sum_{|J|=p, |K|=q} T_{JK}(z)\, dz_J \wedge d\bar{z}_K$$

- The current $T$ is said to be positive if the distribution $\sum \lambda_j \overline{\lambda}_k T_{JK}$ is a positive real measure for all $(\lambda_J) \in \mathbb{C}^N$ (so that $T_{KJ} = \overline{T}_{JK}$, hence $\overline{T} = T$).
- The coefficients $T_{JK}$ are then complex measures – and the diagonal ones $T_{JJ}$ are positive real measures.

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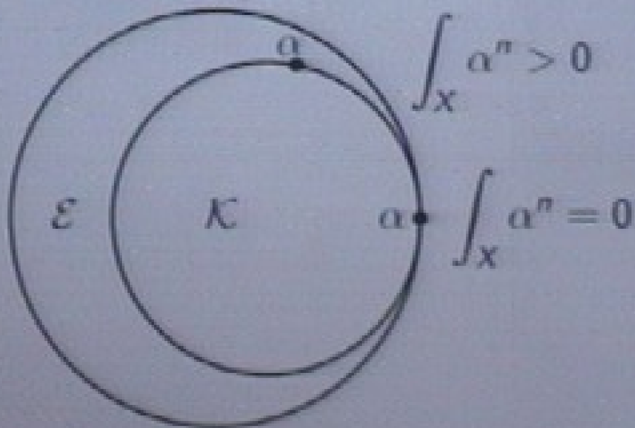

# Final step of proof

Clearly the open cone $\mathcal{K}$ is contained in $\mathcal{P}$, hence in order to show that $\mathcal{K}$ is one of the connected components of $\mathcal{P}$, we need only show that $\mathcal{K}$ is closed in $\mathcal{P}$, i.e. that $\overline{\mathcal{K}} \cap \mathcal{P} \subset \mathcal{K}$. Pick a class $\{\alpha\} \in \overline{\mathcal{K}} \cap \mathcal{P}$. In particular $\{\alpha\}$ is nef and satisfies $\int_X \alpha^n > 0$. Hence $\{\alpha\}$ contains a Kähler current $T$.

Now, an induction on dimension using the assumption $\int_Y \alpha^p > 0$ for all analytic subsets $Y$ (we also use resolution of singularities for $Y$ at this step) shows that the restriction $\{\alpha\}_{|Y}$ is the class of a Kähler current on $Y$.

We conclude that $\{\alpha\}$ is a Kähler class by results of Paun (PhD 1997), therefore $\{\alpha\} \in \mathcal{K}$.

# Variants of the main theorem

**Corollary 1** (DP2004). *Let $X$ be a compact Kähler manifold.*

$$\{\alpha\} \in H^{1,1}(X,\mathbb{R}) \text{ is Kähler} \Leftrightarrow \exists\omega \text{ Kähler s.t. } \int_Y \alpha^k \wedge \omega^{p-k} > 0$$

*for all $Y \subset X$ irreducible and all $k = 1, 2, \ldots, p = \dim Y$.*

*Proof.* Argue with $(1-t)\alpha + t\omega$, $t \in [0,1]$.

**Corollary 2** (DP2004). *Let $X$ be a compact Kähler manifold.*

$$\{\alpha\} \in H^{1,1}(X,\mathbb{R}) \text{ is nef } (\alpha \in \overline{\mathcal{K}}) \Leftrightarrow \forall\omega \text{ Kähler } \int_Y \alpha \wedge \omega^{p-1} \geq 0$$

*for all $Y \subset X$ irreducible and all $k = 1, 2, \ldots, p = \dim Y$.*

**Consequence.** *the dual of the nef cone $\overline{\mathcal{K}}$ is the closed convex cone in $H^{n-1,n-1}_{\mathbb{R}}(X)$ generated by cohomology classes of currents of the form $[Y] \wedge \omega^{p-1}$ in $H^{n-1,n-1}(X,\mathbb{R})$.*

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**Corollary 1** (DP2004). *Let $X$ be a compact Kähler manifold.*

$$\{\alpha\} \in H^{1,1}(X,\mathbb{R}) \text{ is Kähler} \Leftrightarrow \exists\omega \text{ Kähler s.t. } \int_Y \alpha^k \wedge \omega^{p-k} > 0$$

*for all $Y \subset X$ irreducible and all $k = 1, 2, \ldots, p = \dim Y$.*

*Proof.* Argue with $(1-t)\alpha + t\omega$, $t \in [0,1]$.

**Corollary 2** (DP2004). *Let $X$ be a compact Kähler manifold.*

$$\{\alpha\} \in H^{1,1}(X,\mathbb{R}) \text{ is nef } (\alpha \in \overline{\mathcal{K}}) \Leftrightarrow \forall\omega \text{ Kähler } \int_Y \alpha \wedge \omega^{p-1} \geq 0$$

*for all $Y \subset X$ irreducible and all $k = 1, 2, \ldots, p = \dim Y$.*

**Consequence.** *the dual of the nef cone $\overline{\mathcal{K}}$ is the closed convex cone in $H^{n-1,n-1}_{\mathbb{R}}(X)$ generated by cohomology classes of currents of the form $[Y] \wedge \omega^{p-1}$ in $H^{n-1,n-1}(X,\mathbb{R})$.*

# Deformations of compact Kähler manifolds

A deformation of compact complex manifolds is a proper holomorphic map

$$\pi : \mathcal{X} \to S \qquad \text{with smooth fibers } X_t = \pi^{-1}(t).$$

**Basic question** (Kodaira ~ 1960). Is every compact Kähler manifold $X$ a limit of projective manifolds :

$$X \simeq X_0 = \lim X_{t_\nu}, \quad t_\nu \to 0, \quad X_{t_\nu} \text{ projective ?}$$

**Recent results by Claire Voisin (2004)**

- *In any dimension $\geq 4$, $\exists X$ compact Kähler manifold which does not have the homotopy type (or even the homology ring) of a complex projective manifold.*
- *In any dimension $\geq 8$, $\exists X$ compact Kähler manifold such that no compact bimeromorphic model $X'$ of $X$ has the homotopy type of a projective manifold.*

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# Conjecture on deformation stability of the Kähler property

**Theorem** (Kodaira and Spencer 1960).
The Kähler property is open with respect to deformation :
if $X_{t_0}$ is Kähler for some $t_0 \in S$, then the nearby fibers $X_t$ are also Kähler (where "nearby" is in metric topology).

We expect much more.

**Conjecture**. *Let $\mathcal{X} \to S$ be a deformation with irreducible base space $S$ such that some fiber $X_{t_0}$ is Kähler. Then there should exist a countable union of analytic strata $S_\nu \subset S$, $S_\nu \neq S$, such that*

- *$X_t$ is Kähler for $t \in S \setminus \bigcup S_\nu$.*
- *$X_t$ is bimeromorphic to a Kähler manifold* (*i.e. has a Kähler current*) *for $t \in \bigcup S_\nu$.*

# Theorem on deformation stability of Kähler cones

**Theorem** (Demailly-Păun 2004). *Let $\pi : \mathcal{X} \to S$ be a deformation of compact Kähler manifolds over an irreducible base $S$. Then there exists a countable union $S' = \bigcup S_\nu$ of analytic subsets $S_\nu \subsetneq S$, such that the Kähler cones $\mathcal{K}_t \subset H^{1,1}(X_t, \mathbb{C})$ of the fibers $X_t = \pi^{-1}(t)$ are $\nabla^{1,1}$-invariant over $S \setminus S'$ under parallel transport with respect to the $(1,1)$-projection $\nabla^{1,1}$ of the Gauss-Manin connection $\nabla$ in the decomposition of*

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & 0 \\ * & \nabla^{1,1} & * \\ 0 & * & \nabla^{0,2} \end{pmatrix}$$

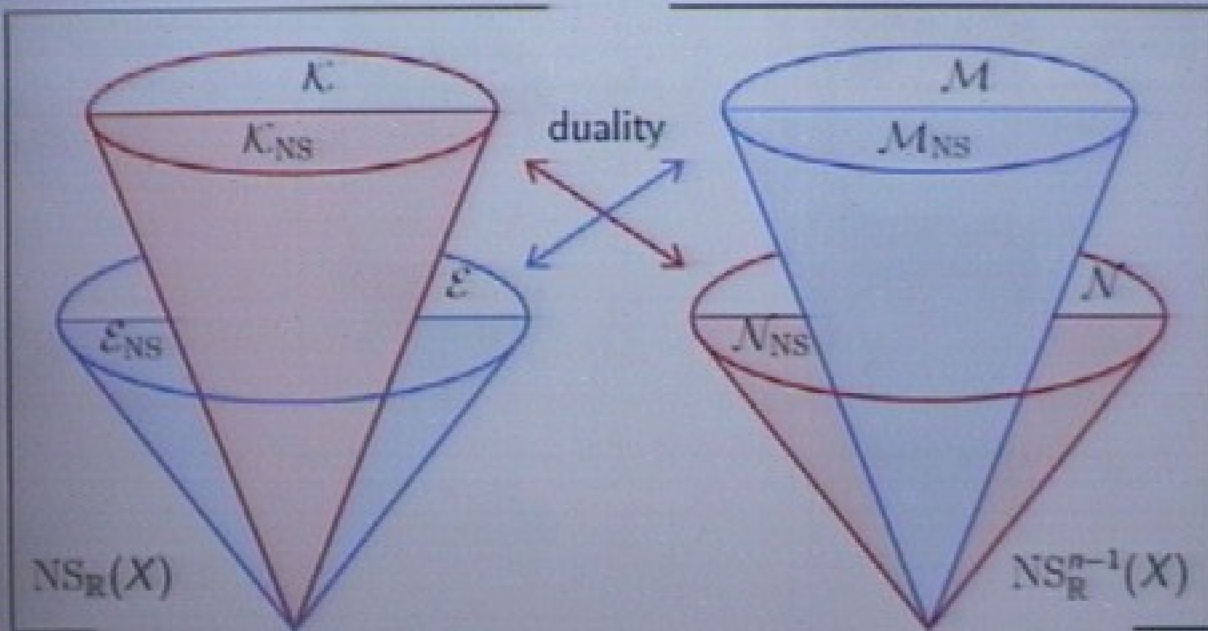
*on the Hodge bundle $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.*

# Positive cones in $H^{n-1,n-1}(X)$ and Serre duality

**Definition.** *Let X be a compact Kähler manifold.*

- *Cone of $(n-1, n-1)$ positive currents*
  $\mathcal{N} = \overline{\mathrm{cone}}\{\{T\} \in H^{n-1,n-1}(X, \mathbb{R});\ T \text{ closed} \geq 0\}$.
- *Cone of effective curves*
  $\mathcal{N}_{\mathrm{NS}} = \mathcal{N} \cap \mathrm{NS}_{\mathbb{R}}^{n-1,n-1}(X)$,
  $= \overline{\mathrm{cone}}\{\{C\} \in H^{n-1,n-1}(X, \mathbb{R});\ C \text{ effective curve}\}$.
- *Cone of movable curves : with $\mu : \widetilde{X} \to X$, let*
  $\mathcal{M}_{\mathrm{NS}} = \overline{\mathrm{cone}}\{\{C\} \in H^{n-1,n-1}(X, \mathbb{R});\ [C] = \mu_*(H_1 \cdots H_{n-1})\}$
  *where $H_j$ = ample hyperplane section of $\widetilde{X}$.*
- *Cone of movable currents : with $\mu : \widetilde{X} \to X$, let*
  $\mathcal{M} = \overline{\mathrm{cone}}\{\{T\} \in H^{n-1,n-1}(X, \mathbb{R});\ T = \mu_*(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})\}$
  *where $\widetilde{\omega}_j$ = Kähler metric on $\widetilde{X}$.*

# Main duality theorem




$H^{1,1}(X,\mathbb{R})$ ← Serre duality → $H^{n-1,n-1}(X,\mathbb{R})$

- The current of integration over a codimension $p$ analytic cycle $A = \sum c_j A_j$ is defined by duality as $[A] = \sum c_j [A_j]$ with

$$\langle [A_j], u \rangle = \int_{A_j} u_{|A_j}$$

for every $(n-p, n-p)$ test form $u$ on $X$.

- Hessian forms of plurisubharmonic functions :

$$\varphi \text{ plurisubharmonic} \Leftrightarrow \left( \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right) \geq 0$$

then

$$T = i\partial\overline{\partial}\varphi \qquad \text{is a closed positive } (1,1)\text{-current.}$$

# Precise duality statement

Recall that the Serre duality pairing is

$$(\alpha^{(p,q)}, \beta^{(n-p,n-q)}) \longmapsto \int_X \alpha \wedge \beta.$$

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**Theorem** (Demailly-Păun 2001)
*If $X$ is compact Kähler, then*
*$\mathcal{K}$ and $\mathcal{N}$ are dual cones.*
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**Theorem** (Boucksom-Demailly-Paun-Peternell 2004)
*If $X$ is projective algebraic, then*
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**Conjecture** (Boucksom-Demailly-Paun-Peternell 2004)
*If $X$ is Kähler, then*
*$\mathcal{E}$ and $\mathcal{M}$ should be dual cones.*

# Concept of volume (very important !)

**Definition** (Boucksom 2002).
*The volume (movable self-intersection) of a big class $\alpha \in \mathcal{E}^\circ$ is*

$$\mathrm{Vol}(\alpha) = \sup_{T \in \alpha} \int_{\widetilde{X}} \beta^n > 0$$

*where the supremum is taken over all Kähler currents $T \in \alpha$ with logarithmic poles, and $\mu^* T = [E] + \beta$ with respect to some modification $\mu : \widetilde{X} \to X$.*

If $\alpha \in \mathcal{K}$, then $\mathrm{Vol}(\alpha) = \alpha^n = \int_X \alpha^n$.

**Theorem.** (Boucksom 2002). *If $L$ is a big line bundle and*
$\mu_m^*(mL) = [E_m] + [D_m]$
*(where $E_m$ = fixed part, $D_m$ = moving part), then*

$$\mathrm{Vol}(c_1(L)) = \lim_{m \to +\infty} \frac{n!}{m^n} h^0(X, mL) = \lim_{m \to +\infty} D_m^n.$$

# Movable intersection theory

**Theorem** (Boucksom 2002) *Let $X$ be a compact Kähler manifold and*

$$H^{k,k}_{\geq 0}(X) = \{\{T\} \in H^{k,k}(X, \mathbb{R});\ T \text{ closed} \geq 0\}.$$

- $\forall k = 1, 2, \ldots, n$, $\exists$ *canonical "movable intersection product"*

  $$\mathcal{E} \times \cdots \times \mathcal{E} \to H^{k,k}_{\geq 0}(X), \quad (\alpha_1, \ldots, \alpha_k) \mapsto \langle \alpha_1 \cdot \alpha_2 \cdots \alpha_{k-1} \cdot \alpha_k \rangle$$

  *such that* $\mathrm{Vol}(\alpha) = \langle \alpha^n \rangle$ *whenever* $\alpha$ *is a big class.*
- *The product is increasing, homogeneous of degree 1 and superadditive in each argument, i.e.*

  $$\langle \alpha_1 \cdots (\alpha_j' + \alpha_j'') \cdots \alpha_k \rangle \geq \langle \alpha_1 \cdots \alpha_j' \cdots \alpha_k \rangle + \langle \alpha_1 \cdots \alpha_j'' \cdots \alpha_k \rangle.$$

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  *It coincides with the ordinary intersection product when the $\alpha_j \in \overline{\mathcal{K}}$ are nef classes.*

# Construction of the movable intersection product

First assume that all classes $\alpha_j$ are big, i.e. $\alpha_j \in \mathcal{E}^\circ$. Fix a smooth closed $(n-k, n-k)$ *semi-positive* form $u$ on $X$. We select Kähler currents $T_j \in \alpha_j$ with logarithmic poles, and simultaneous more and more accurate log-resolutions $\mu_m : \widetilde{X}_m \to X$ such that

$$\mu_m^* T_j = [E_{j,m}] + \beta_{j,m}.$$

We define

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as a weakly convergent subsequence. The main point is to show that there is actually convergence and that the limit is unique in cohomology ; this is based on "monotonicity properties" of the Zariski decomposition.

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## Generalized abundance conjecture

**Definition.** *For a class $\alpha \in H^{1,1}(X,\mathbb{R})$, the numerical dimension $\nu(\alpha)$ is*

- $\nu(\alpha) = -\infty$ *if $\alpha$ is not pseudo-effective,*
- $\nu(\alpha) = \max\{p \in \mathbb{N}\,;\ \langle\alpha^p\rangle \neq 0\} \quad \in \{0, 1, \ldots, n\}$ *if $\alpha$ is pseudo-effective.*

**Conjecture** ("generalized abundance conjecture"). *For an arbitrary compact Kähler manifold X, the Kodaira dimension should be equal to the numerical dimension :*

$$\kappa(X) = \nu(c_1(K_X)).$$

**Remark.** The generalized abundance conjecture holds true when $\nu(c_1(K_X)) = -\infty, 0, n$ (cases $-\infty$ and $n$ being easy).

- The current of integration over a codimension $p$ analytic cycle $A = \sum c_j A_j$ is defined by duality as $[A] = \sum c_j [A_j]$ with

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# Proof of duality between $\mathcal{E}_{NS}$ and $\mathcal{M}_{NS}$

**Theorem** (Boucksom-Demailly-Păun-Peternell 2004).
For $X$ projective, a class $\alpha$ is in $\mathcal{E}_{NS}$ (pseudo-effective) if and only if it is dual to the cone $\mathcal{M}_{NS}$ of moving curves.

*Proof of the theorem.* We want to show that $\mathcal{E}_{NS} = \mathcal{M}_{NS}^{\vee}$. By obvious positivity of the integral pairing, one has in any case

$$\mathcal{E}_{NS} \subset (\mathcal{M}_{NS})^{\vee}.$$

If the inclusion is strict, there is an element $\alpha \in \partial\mathcal{E}_{NS}$ on the boundary of $\mathcal{E}_{NS}$ which is in the interior of $\mathcal{N}_{NS}^{\vee}$. Hence

$$(*) \qquad \alpha \cdot \Gamma \geq \varepsilon\omega \cdot \Gamma$$

for every moving curve $\Gamma$, while $\langle \alpha^n \rangle (= \mathrm{Vol}(\alpha)) = 0$.

# Characterization of uniruled varieties

Recall that a projective variety is called uniruled if it can be covered by a family of rational curves $C_t \simeq \mathbb{P}^1_{\mathbb{C}}$.

**Theorem** (Boucksom-Demailly-Paun-Peternell 2004)
*A projective manifold $X$ is not uniruled if and only if $K_X$ is pseudo-effective, i.e. $K_X \in \mathcal{E}_{\mathrm{NS}}$.*

*Proof (of the non trivial implication).* If $K_X \notin \mathcal{E}_{\mathrm{NS}}$, the duality pairing shows that there is a moving curve $C_t$ such that $K_X \cdot C_t < 0$. The standard "bend-and-break" lemma of Mori then implies that there is family $\Gamma_t$ of rational curves with $K_X \cdot \Gamma_t < 0$, so $X$ is uniruled.

# Complex manifolds / Kähler metrics

- A Kähler metric is a smooth positive definite (1,1)-form

$$\omega(z) = i \sum_{1\le j,k\le n} \omega_{jk}(z) dz_j \wedge d\overline{z}_k \qquad \text{such that } d\omega = 0.$$

- The manifold $X$ is said to be Kähler (or of Kähler type) if it possesses at least one Kähler metric $\omega$ [Kähler 1933]
- Every complex analytic and locally closed submanifold $X \subset \mathbb{P}^N_{\mathbb{C}}$ in projective space is Kähler when equipped with the restriction of the Fubini-Study metric

$$\omega_{FS} = \frac{i}{2\pi} \log(|z_0|^2 + |z_1|^2 + \ldots + |z_N|^2).$$

- Especially projective algebraic varieties are Kähler.

# Sheaf / De Rham / Dolbeault / cohomology

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- Sheaf cohomology $H^q(X, \mathcal{F})$
  especially when $\mathcal{F}$ is a coherent analytic sheaf.
- Special case : cohomology groups $H^q(X, R)$ with values in constant coefficient sheaves $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$
  = De Rham cohomology groups.