Computation

- Most scientific problems do not have a closed form solution
- To solve them we must resort to computation and create an approximate solution
- To do this we create a numerical algorithm
- There are typically a myriad of algorithms
- Is there a way to tell whether algorithm is optimal or near optimal?
- How do we define optimal?
- Computations are not exact
- We cant even enter a real number into a computer without some error
- Numerical Stability

Discrete Sensing

- $x \in I\!R^N$ with N large
- We are able to ask n non-adaptive questions about x
- Question means inner product $v \cdot x$ with $v \in I\!R^N$ - called sample
- Any such sampling is given by is an $n \times N$ matrix Φ : the entries in $y = \Phi x$ are the answers to our questions

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- We can not prove high performance for deterministic constructions
- Deterministic constructions based on coding/finite fields etc. give $k \leq C_0\sqrt{n}$

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- It is not yet clear exactly how these demands play against one another
- Theoretical Computer Science: Gilbert, Muthukrishnan, Strauss, and many others

Analog Systems

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- Web site on Compressed Sensing: Rice ECE

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- Two issues: (i) Enough information in y; (ii) How to extract this information: Decoder

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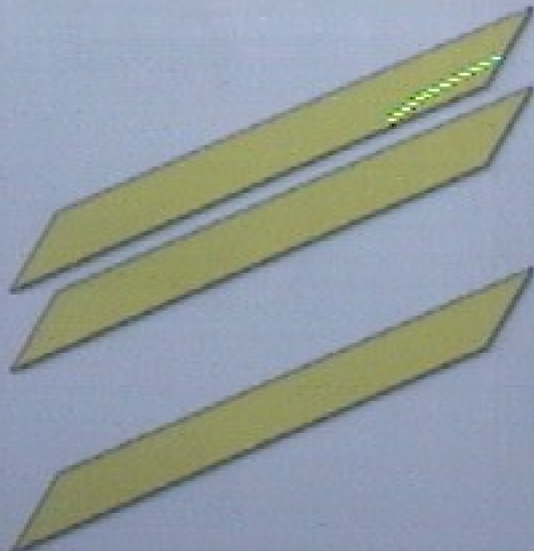

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- Answer $n = 2k$

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Theorem: If Φ is any $n \times N$ matrix and $2k \le n$, then the following are equivalent:

(i) There is a Δ such that $\Delta(\Phi(x)) = x$, for all $x \in \Sigma_k$,
(ii) $\Sigma_{2k} \cap \mathcal{N}(\Phi) = \{0\}$,
(iii) For any set T with $\#T = 2k$, the matrix Φ_T has rank $2k$.

(iv) For any set T with $\#T = 2k$, the $2k \times 2k$ matrix $\Phi_T^* \Phi_T$ is invertible: all its eigenvalues are positive.

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- Given k can we construct matrices Φ of size $2k \times N$ with the properties of the theorem?

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- We need N vectors in $\mathbb{R}^{2k}$ such that any $2k$ of them are linearly independent
- Vandermonde matrix. Choose $x_1 < x_2 < \cdots < x_N$

$$\Phi := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{2k-1} & x_2^{2k-1} & \cdots & x_N^{2k-1} \end{pmatrix}$$

Naive Decoding

$$\Delta(y) := \operatorname*{Argmin}_{z \in \Sigma_k} \|y - \Phi(z)\|_{\ell_2}$$

- $X_T := \{z : \ \mathrm{supp}(z) \subset T\}$
- $x_T := \operatorname*{Argmin}_{z \in X_T} \|y - \Phi z\|_{\ell_2} \rightarrow x_T = [\Phi_T^* \Phi_T]^{-1} \Phi_T y$
- $T^* := \mathrm{Argmin}_{\#(T)=k} \|y - \Phi(x_T)\|_{\ell_2}$
- $\Delta(y) := x_{T^*}$

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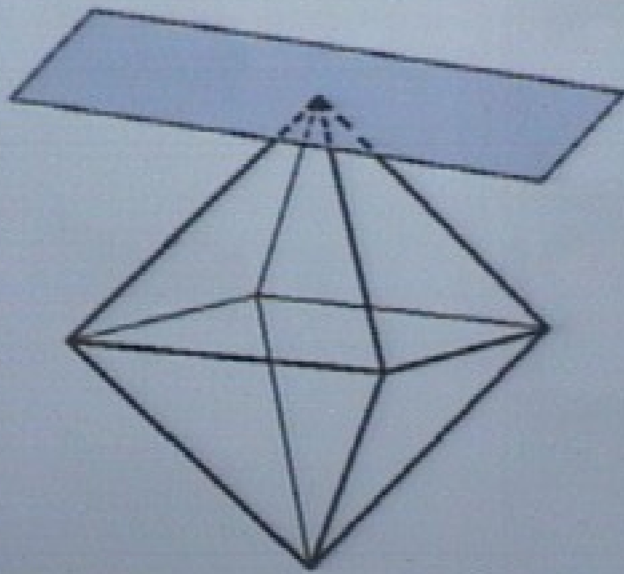

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Lazy man verification of RIP

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- Here each entry $\phi_{i,j}$ is an independent realization of some fixed random variable r with mean zero and variance $1/n$
- Trivially we have $E(\|\Phi(\omega)x\|_{\ell_2^n}^2) = \|x\|_{\ell_2^N}^2$

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Lazy man Proof

Theorem (Baraniuk, Davenport, DeVore, Wakin)
Given c_0 there is a $c_1 > 0$ such that with probability $1 - e^{-c_1 n}$, the matrix $\Phi(\omega)$ satisfies RIP of order k for all $k \leq c_0 n / \log(N/n)$

- We find a net of points $\mathcal{P}$ which cover the unit sphere in Σ_k to accuracy $\delta/4$
- Using the concentration inequality, we see that with high probability the draw of $\Phi = \Phi(\omega)$ satisfies

$$(1-\delta/2)\|q\|_{\ell_2^N} \leq \|\Phi(q)\|_{\ell_2^n} \leq (1+\delta/2)\|q\|_{\ell_2^N}, \quad q \in \mathcal{P}$$

- Extend this to all $x \in \Sigma_k$ by a boot strapping estimate: Given x with $\|x\|_{\ell_2^N} = 1$ find q such that $\|x - q\|_{\ell_2^N} \leq \delta/4$

$$\|\Phi(x)\|_{\ell_2^n} \leq \|\Phi(x-q)\|_{\ell_2^n} + \|\Phi(q)\|_{\ell_2^n} \leq M\delta/4 + 1 + \delta/2$$

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Universality

- Suppose we have a collection of basis $\mathcal{B} := \{B\}$
- $\#(\mathcal{B}) \leq Ce^{cn}$
- These probability arguments give the existence of a matrix Φ which satisfies RIP of order k simultaneously for all $B \in \mathcal{B}$ and $k \leq cn/\log(N/n)$
- This means this one Φ will capture sparsity with respect to any and all of these bases
- However to decode we need to know the basis

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- $x \in \mathbb{R}^N$ with N large
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Model II: Compressible Signals

- The sparse signal classes do not represent real signals: signals will typically have all entries nonzero but most will be small
- A compressible signal x is one that can be approximated well by elements from Σ_k:
  $\sigma_k(x)_X := \inf_{z \in \Sigma_k} \|x - z\|_X$
- Typical signal classes are $U(\ell_p^N)$ and typical $X = \ell_q^N$
  $\|x\|_{\ell_p^N} := \left(\sum_{i=1}^N |x_i|^p\right)^{1/p}$
- If $x \in U(\ell_p^N)$ then $\sigma_k(x)_{\ell_q^N} \le k^{1/q-1/p}, \quad p < q$
- Example ($q = 2$, $p = 1$): $\sigma_k(x)_{\ell_2^N} \le k^{-1/2}$

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Optimality on Models II

- Let K be a compact set in $\mathbb{R}^N$
- Best performance on the class K in norm $\|\cdot\|_X$

$$E_n(K)_X := \inf_{(\Phi,\Delta)\in A_n} \sup_{x\in K} \|x - \Delta(\Phi(x))\|_X$$

- Here $A_n = \{(\Phi, \Delta) : \Phi \text{ is } n \times N\}$
- The pair (Φ, Δ) is near optimal if

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Optimality on ℓ_p^N classes

- The asymptotic behavior of $E_n(K)_X$ is known for all $K = U(\ell_p^N)$ in all $X = \ell_q^N$
- Solved in 1970's and 1980's in Approximation Theory and Finite Dimensional Geometry
- Kashin, Gluskin main players
- Example ($p = 1, q = 2$)

$$C_0\sqrt{\frac{\log(N/n)}{n}} \leq E_n(U(\ell_1^N))_{\ell_2^N} \leq C_1\sqrt{\frac{\log(N/n)}{n}}$$

- These results do not provide practical encoding/decoding schemes
- Candes-Tao , Donoho show that RIP + ℓ_1 minimization give near optimal performance for $q = 2,\ p \leq 1$

Instance-Optimal

- We say (Φ, Δ) is Instance-Optimal of order k for X if for an absolute constant $C > 0$ (independent of k, n, N)

$$\|x - \Delta(\Phi(x))\|_X \leq C\sigma_k(x)_X$$

- We will be interested in $X = \ell_q^N$
- Problem: for a given X and size $n \times N$ find the largest values of k for which we have instance-optimality and the encoder-decoder pairs (Φ, Δ) which admit these values of k
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Good News

- Let $X = \ell_1^N$ and let Φ satisfy RIP for $3k$, i.e. $\delta_{3k} < 1$ then there is a decoder such that (Φ, Δ) is instance optimal for k:

$$\|x - \Delta(\Phi(x))\|_{\ell_1^N} \leq C_0 \sigma_k(x)_{\ell_1^N}$$

- Given n we can have instance optimality if $k \leq c_0 n / \log(N/n)$
- Bonus: Decoding can be done by ℓ_1 Minimization
- Although not explicitly stated there this result is easily derived from the work of Candes-Tao

Discrete Sensing

- $x \in \mathbb{R}^N$ with N large
- We are able to ask n non-adaptive questions about x
- Question means inner product $v \cdot x$ with $v \in \mathbb{R}^N$ - called sample

Bad News

- Let $X = \ell_2^N$, then in order to have instance optimality for $k = 1$ we need $n \geq c_0 N$
- Here c_0 depends on the instance optimality constant C
- OOPS: Instance Optimality is not a Viable Concept in ℓ_2^N

Instance-Optimality in Probability

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- Suppose $\Phi(\omega)$ is a collection of random matrices
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- We say $\{\Phi(\omega)\}$ is bounded with probability $1-\epsilon$ if given any $x \in \mathbb{R}^N$ with probability $1-\epsilon$ a random draw $\{\Phi(\omega)\}$ will satisfy

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with C_0 an absolute constant

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Theorem: Cohen-Dahmen-DeVore

- If $\{\Phi(\omega)\}$ satisfies RIP of order $3k$ and boundedness each with probability $1-\epsilon$ then there are decoders $\Delta(\omega)$ such that given any $x \in \ell_2^N$ we have with probability $1-2\epsilon$

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- Instance-optimality in probability
- Range of k is $k \leq c_0 n / \log(N/n)$
- Decoder is the least squares minimization

Optimal Matrices

- We have only built Optimal Matrices using probabilistic methods