Symplectic Field Theory (SFT)

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Thanks to the "master equation" $[\mathbf{H}, \mathbf{H}] = 0$, we can define on W a differential $D_{\mathbf{H}}(A) = [A, H], A \in W$, which satisfies $D_{\mathbf{H}}^2 = 0$. In many cases the homology $H_*(W, D_{\mathbf{H}})$ provides us with a powerful geometric invariant.

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- In the Floer case, this gives a far-going generalization of the Floer homology theory, by bringing to it new invariant algebraic structures.
- In the contact case this leads to the contact homology theory, which, for contact manifolds of dimension > 3, is essentially the only known source of invariants.

Alternatively, one may use $\mathbf{H}$ to define a differential $d_{\mathbf{H}}$ on the Fock space Fock $= \{\mathbf{g} = \sum_{k \geq 0} g_k(q)\hbar^k\}$:

$$d_{\mathbf{H}}(\mathbf{g}) = \overrightarrow{\mathbf{H}}\mathbf{g},$$

which leads to a structure of a BV_∞-algebra on Fock. This formalism was recently explored by Cieliebak-Latchev.

Let (X, ω) be a symplectic cobordism which bounds (on its positive end) a Hamiltonian structure $\mathcal{H} = (\omega, \lambda)$ on $Y = \partial X$, and J a compatible almost complex structure on Y.

As it was explained above, the "naive" Rational SFT associates with (X, ω, J) a Lagrangian submanifold $L_{\mathbf{f}} \subset \mathbf{S}$ generated by a function $\mathbf{f}(p)$, such that the Hamiltonian $\mathbf{h}(q, p)$ vanishes on $L_{\mathbf{f}}$.

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In a little bit more advanced version of Rational SFT, in a way similar to how this was done above for the Hamiltonian, we associate with a sequence $\Theta = (\theta_1, \dots, \theta_k)$ of differential forms on X a function

$$\mathbf{f}^{\Theta}(T, q, p) \in \mathbf{P} \otimes \mathbf{C}[T],$$

where $\mathbf{C}[T]$ is a graded polynomial algebra generated by graded variables $T = (t_{ij}), i = 1, \dots, k; j \geq 0$.

If $d\Theta = 0$ then similarly to the "naive case", $\mathbf{h}^{\Theta|_Y}$ vanishes on $L_{\mathbf{f}^{\Theta}}$.

Take now Θ with $d\Theta|_Y = 0$. Then we have

$$\mathbf{h}^{\Theta_Y}(T)\big|_{L_{\mathbf{f}^{\Theta \cup d\Theta}(T,S)}} = D\mathbf{f}^{\Theta \cup d\Theta}(T,S), \qquad (8)$$

where $D = \sum_{i,j} t_{ij} \frac{\partial}{\partial s_{ij}}$ and $S = (s_{ij})$ are variables associated to $d\Theta = (d\theta_1, \dots, d\theta_k)$.

Take, for instance, $Y = S^3$, X is a 4-ball, $\Theta = (\theta)$ where θ is a 3-form which restricts to the standard volume form on S^3. Then, by differentiating (8) with respect to $t = t_{10}$ and setting $T = 0$ (and denoting $\mathbf{f} = \mathbf{f}^{\Theta}$, $s = s_{10}$), we get

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$$\frac{\partial \mathbf{f}}{\partial s}(S) = \frac{\partial \mathbf{h}}{\partial t}(0)\big|_{L_{\mathbf{f}(S)}},$$

which is the Hamilton-Jacobi equation for the evolution of the Lagrangian manifold L_s, $s = s_{10}$, under the flow of the Hamiltonian $\mathbf{h}_1 = \frac{\partial \mathbf{h}^{\theta}(T)}{\partial t}(0)$.

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The Lie algebra $\mathfrak{d}$ of $\mathfrak{D}$ consists of symplectic vector fields, i.e. tangent vector fields $v \in TM$ such that the form $\alpha = i(v)\omega$ is closed. If α is exact, $\alpha = dH$, then the vector field $v = \mathrm{sgrad} H$ is called Hamiltonian, and the function H its Hamiltonian function.

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In the Darboux coordinates, we have $\mathrm{sgrad} H = \left(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)$. The function H is determined by $\mathrm{sgrad} H$ uniquely up to an additive constant.

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- Given a symplectic manifold (X, ω) a map $f : X \to X$ is *symplectic*, i.e. $f^*\omega = \omega$, if and only if its graph $\Gamma_f \subset X \times X$ is Lagrangian with respect to the symplectic form $\Omega = \omega \times (-\omega)$ on $X \times X$.

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- the total volume (i.e. $\int\limits_X \omega^n$);
- the homotopy class of a compatible almost complex structure J, and
- the cohomology class $[\omega] \in H^2(X)$, in the case of a closed manifold X.

It was great Gromov's insight, when he introduced holomorphic curves as a tool for finding more subtle, specifically symplectic invariants.

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C. Abbas, F. Bourgeois, K. Cieliebak, T. Coates, T. Ekholm, J. Etnyre, R. Hind, A. Ivrii, D. McDuff, E. Katz, S.-S. Kim, J. Latchev, L. Ng, K. Mohnke, K. Wysocki, B. Parker, R. Siefring, J. Sabloff, M. Sullivan, M.-L. Yau, I. Ustilovsky, E. Zehnder,. . . .

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- If $\dim S > 2$ the system (1) is overdetermined, and usually has no solutions, even locally.
- When $\dim S = 2$, i.e. when (S, J_S) is a Riemann surface then the system (1) is determined, and regardless of integrability of J_X, the local theory of holomorphic maps $S \to X$, or as they called in this case, holomorphic (also J-holomorphic, pseudo-holomorphic) curves, is as rich as in the integrable case.

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Thus, under certain transversality assumptions the moduli spaces of holomorphic curves form finite dimensional manifolds, or at least orbifolds.

Gromov compactness theorem for holomorphic curves provides a compactification of the moduli spaces of holomorphic curves, similar to the Knudsen-Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points.

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$$(S_g, j, x_1, \dots, x_n)$$

of conformal structures on a closed surface g with n marked points, can be identified for $2g + n \geq 3$, with the moduli space of complete hyberbolic metrics of finite area on $S_g \setminus \{x_1, \dots, x_n\}$.

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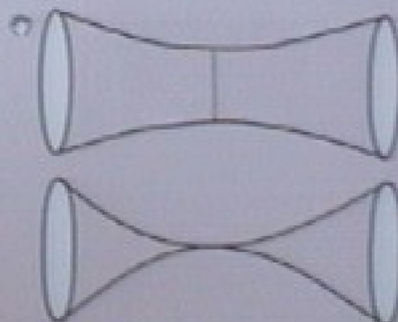
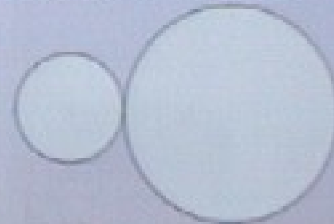


The compactification of the moduli space of holomorphic curves provided by Gromov compactness theorem (in Kontsevich's interpretation) differs only in the stability condition: it needs to be satisfied only for constant components of nodal curves (which are sometimes called ghosts).

In particular one may have a phenomenon of bubbling off of holomorphic spheres.



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We also benefited a lot from discussions and collaboration with the people working on tightly related subjects, (e.g. relative Gromov-Witten theory, Cluster Homology, String Topology etc.).

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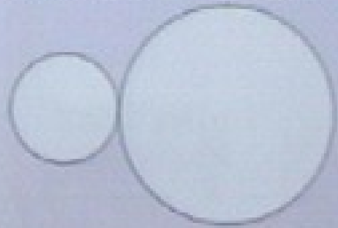

The role of a compatible with J symplectic form ω is to ensure the area bounds for J-holomorphic curves. Indeed, the area of any holomorphic curve $f : S \to X$ coincides with its symplectic area $\int_S f^*\omega$, and hence for a closed S the area of a holomorphic curve $f : S \to X$ depends only on its homology class in $H_2(X)$.

Gromov's scheme of defining symplectic invariants via the holomorphic curve theory was the following:

- Pick a (generic) almost complex structure J compatible with ω.
- Then, the compactified moduli space $\overline{\mathcal{M}}_g^A(X, J)$ of J-holomorphic curves of genus g in a fixed homology class $A \in H_2(X)$ form a cycle in the similarly enriched moduli space $\overline{\mathcal{F}}_g^A(X)$ of all smooth maps $F : (S_g, j) \to X$.

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- The compactness theorem ensures that the homology class of this cycle remains unchanged when one varies J, while keeping it compatible with ω.
- But all J, which are compatible with a fixed ω, are homotopic, and hence the homology class of $\overline{\mathcal{M}}_g^A(X, J)$ in $\overline{\mathcal{F}}_g^A(X)$, called Gromov invariant, depends only on ω.

Theorem (Gromov)

Take $\mathbb{R}^3 = \{p_2 = 0\} \subset \mathbb{R}^4$. Let B be the unit ball $\{p_1^2 + q_1^2 + q_2^2 < 1\} \subset \mathbb{R}^3$ and set $H = \mathbb{R}^3 \setminus B$. Let $h_\pm : B^4(R) \to \mathbb{R}^4_\pm$ be two orthogonal embeddings of the 4-ball of radius $R > 1$ into the upper- and lower-half spaces $\mathbb{R}^4_+ = \{p_2 > 0\}$ and $\mathbb{R}^4_- = \{p_2 < 0\}$. Then h_- and h_+ are not symplectically isotopic in $\mathbb{R}^4 \setminus H$.

Example: Symplectic Camel Problem

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Suppose there exists a symplectic isotopy $h_t : B^4(R) \to \mathbb{R}^4 \setminus H$ connecting $h_0 = h_+$ and $h_1 = h_-$. For each $t \in [0,1]$, choose an almost complex structure J_t on $\mathbb{R}^4$, such that

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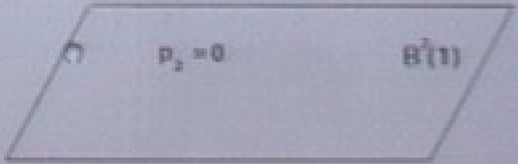

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Let $\mathcal{M}_t$ be the moduli space of J_t-holomorphic discs with 1 marked point in $\mathbb{R}^4$ with boundary in L.

The evaluation map $ev : \mathcal{M}_t \to \mathbb{R}^4$ provides a smooth hyperface Σ_t bounded by L, and center of the moving ball $h_t(B^4(R))$ **should hit Σ_t at a certain moment $t = t_0$.**

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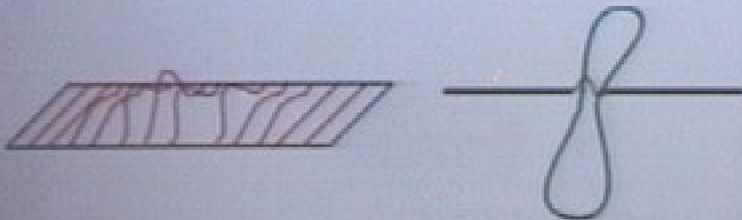

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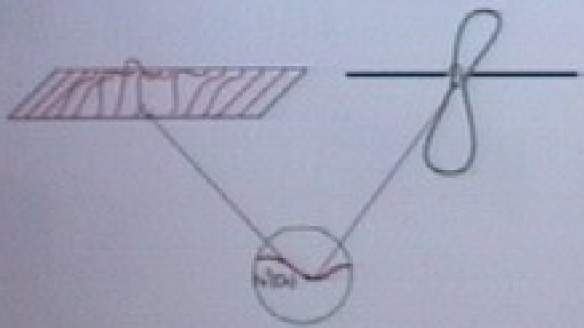

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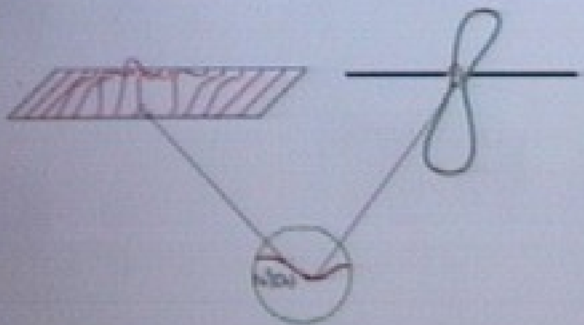

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On the other hand,

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$$\langle \theta_1, \dots, \theta_k \rangle_{g,k}^{A} = \int\limits_{\overline{\mathcal{M}}_{g,k}^{A}(X,J)} \mathrm{ev}_1^*(\theta_1) \wedge \cdots \wedge \mathrm{ev}_k^*(\theta_k). \tag{2}$$

where

Gromov-Witten potential is a certain, motivated by Physics way to package algebraic information contained in the Gromov invariant. Let (X, ω, J) be a symplectic manifold with a compatible almost complex structure J. Given cohomology classes $\theta_1, \dots, \theta_k \in H^*(X)$ we define their **correlator**

$$\langle \theta_1, \dots, \theta_k \rangle_{g,k}^A = \int_{\overline{\mathcal{M}}_{g,k}^A(X,J)} ev_1^*(\theta_1) \wedge \dots \wedge ev_k^*(\theta_k). \qquad (2)$$

where

- $\overline{\mathcal{M}}_{g,k}^A(X,J)$ is the compactified moduli space of J-holomorphic curves of genus g with k marked points $x_1, \dots, x_k \in S_g$ in a given homology class $A \in H_2(X)$;
- $ev_i : \overline{\mathcal{M}}_{g,n}^A(X,J) \to X$ is the evaluation map at the point x_i, $i = 1, \dots, k$.

Let us fix a basis $\theta_1, \ldots, \theta_N \in H^*(X)$ and a basis $A_1, \ldots, A_K \in H^2(X)$. Let $(t_1, \ldots, t_N)$ be the graded coordinates in $H^*(X)$ in the chosen basis. We will also identify a homology class $A \in H_2(X)$ with its degree $d = (d_1, \ldots, d_K)$, i.e. coordinates in the basis $A_1, \ldots, A_m$. Let us also introduce (even graded) variables $z_1, \ldots z_K$ and $\hbar$, and write $z^d = z_1^{d_1} \ldots z_K^{d_K}$.

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Then the Gromov-Witten potential $F(t_0, \ldots, t_N, z)$ is defined by the formula

$$F(t_0, \ldots, t_N, z) = \sum_{g,d,k} \frac{1}{k!} \langle \mathbf{t}, \overset{k}{\ldots}, \mathbf{t} \rangle_{g,k}^{d} z^d \hbar^{g-1}, \tag{3}$$

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One can further upgrade the Gromov-Witten potential to the so-called descendent potential and make it dependent on infinitely many variables $T = (t_{ij})$, $i = 1, \dots, k$, $j = 0, 1, \dots$:

$$F(T, z) = \sum_{g,d,k} \frac{1}{k!} (T, \dots, T)^d_{g,k} z^d \hbar^{g-1}. \tag{4}$$

where

$$(T, \dots, T)^A_{g,k} = \int\limits_{\overline{\mathcal{M}}^A_{g,k}(X,J)} \bigwedge_{m=1}^{k} \left(\sum_{i \in \{1,\dots,k\}, j \geq 0} t_{ij} \, ev_i^*(\theta_i) (c_1(L_m))^j \right).$$

where L_m is the tautological line bundle over $\overline{\mathcal{M}}_{g,m}(X,J)$ formed by cotangent lines at the marked point x_m.

Symplectic Geometry is the geometry of a non-degenerate skew-symmetric bilinear form. Such form exists only on an even-dimensional space, and is unique up to an isomorphism. Viewed as a differential form, it can be written in $\mathbb{R}^{2n} = \{p_1, q_1, \ldots, p_n, q_n\}$ as

$$\sum_1^n dp_j \wedge dq_j = dp \wedge dq.$$

The genus 0 part $F^{(0)}(t_0, \ldots, t_N, z)$ of $F = \sum_{g=0} F^{(g)}\hbar^{g-1}$, is called the rational Gromov-Witten potential.

For instance, for $X = CP^2$ let us choose the standard basis $\theta_0 = 1, \theta_1 \in H^2(CP^2), \theta_2 \in H^4(CP^2)$ and choose $[CP_1]$ as a generator of $H_2(CP_2)$. Consider the function

$$f(t,z) = F^{(0)}(0,0,t,z) = \sum_{d=1}^{\infty}\sum_{k=1}^{\infty} N_{d,k} t^k z^d$$

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For instance, for $X = \mathbb{C}P^2$ let us choose the standard basis $\theta_0 = 1, \theta_1 \in H^2(\mathbb{C}P^2), \theta_2 \in H^4(\mathbb{C}P^2)$ and choose $[\mathbb{C}P_1]$ as a generator of $H_2(\mathbb{C}P_2)$. Consider the function

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$N_{1,2}=1, N_{2,5}=1, N_{3,8}=12, N_{4,11}=620,\dots$ In general, $k=3d-1$ if $N_{d,k}\neq 0$.

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$N_{1,2} = 1, N_{2,5} = 1, N_{3,8} = 12, N_{4,11} = 620, \ldots$. In general, $k = 3d - 1$ if $N_{d,k} \neq 0$. Today several recursion relations (beginning with Kontsevich's) are known for computing $N_{d,k}$.

We will describe here a more geometric recipe for computing the function $f(t, z)$, which is provided by Symplectic Field Theory.

Consider the space S of $\mathbb{C}^2$-valued formal Fourier series

$$U(x) = (u_0(x), u_1(x)) = \sum_{m=1}^{\infty} P_m e^{imx} + Q_m e^{-imx},$$

where $P_m = (p_{m,0}, p_{m,1})$, $Q_m = (q_{m,0}, q_{m,1})$.

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where $P_m = (p_{m,0}, p_{m,1})$, $Q_m = (q_{m,0}, q_{m,1})$. We endow S with a symplectic form $\Omega = \sum \frac{1}{m}(dp_{m,0} \wedge dq_{m,1} + dp_{m,1} \wedge dq_{m,0})$. It corresponds to a Poisson tensor

$$P(\delta U, \delta V) = \frac{1}{2\pi i} \int_0^{2\pi} (\delta U, \delta V') dx,$$

where $(\cdot, \cdot)$ is the inner product with the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, i.e. $(A, B) = a_0 b_1 + a_1 b_0$ for $A = (a_0, a_1)$, $B = (b_0, b_1) \in \mathbb{C}^2$.

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Consider a Hamiltonian function H on S defined by the formula

$$H(U) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{u_0^2(x)}{2} + e^{u_1(x)-ix}\right)dx, \text{ where } U = (u_0, u_1).$$

The flow Φ^t of this Hamiltonian is given by differential equations

$$\dot{u}_0 = -i\frac{d}{dx}\left(e^{u_1(x)-ix}\right);$$

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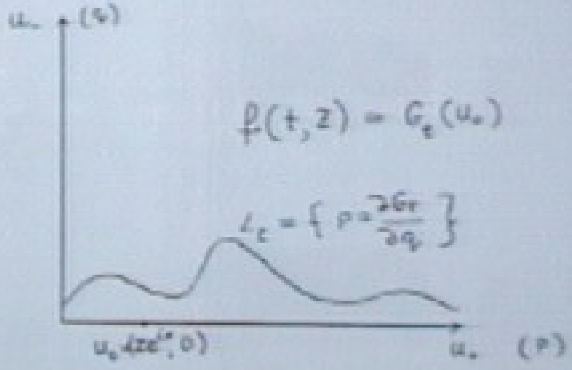
Note that this is a dispersionless Toda equation which can be rewritten as

$$\ddot{u}_1 = -\left(e^{u_1-ix}\right)_{xx}.$$

(I learned this from B. Dubrovin.)

Denote by L_t the image $\Phi^t(L)$ of the Lagrangian subspace $L=\{Q=0\}$ under the Hamiltonian flow Φ^t. If L_t is graphical with respect to the projection $(P,Q)\mapsto P$ (and in the world of formal power series everything is graphical!) then it is defined by a generating function $G_t(P)$.



$$L_t=\begin{cases} q_{m,0} &= m\frac{\partial G_t}{\partial p_{m,1}};\\ q_{m,1} &= m\frac{\partial G_t}{\partial p_{m,0}}.\end{cases}$$

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$$\sum_1^n dp_j \wedge dq_j = dp \wedge dq.$$

If one identifies $\mathbb{R}^{2n} = \{p_1, q_1, \dots, p_n, q_n\}$ with $\mathbb{C}^n = \{z_1 = p_1 + iq_1, \dots, p_n + iq_n\}$, then for any two vectors $V_1, V_2 \in \mathbb{R}^{2n}$ we have $\omega_0(V_1, V_2) = \langle V_1, iV_2 \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{2n}$.
Equivalently, $-\omega$ is the imaginary part of the standard Hermitian product on $\mathbb{C}^n$.

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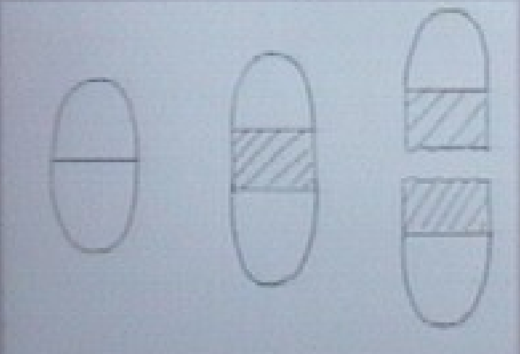
Splitting of a manifold



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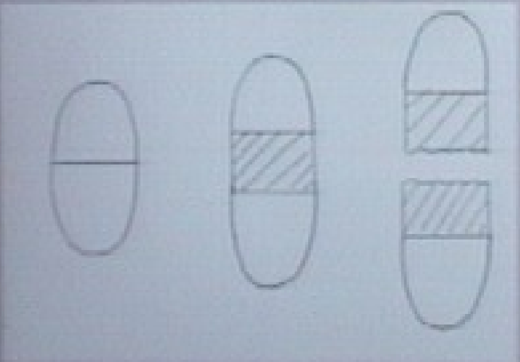


Splitting of the moduli spaces

Cylindrical almost complex manifolds.

An almost complex structure J on $\mathbb{R} \times Y$ is called *cylindrical* if it is invariant under translations

$$(t, x) \mapsto (t + c, y) \ , t, c \in \mathbb{R}, \ y \in Y,$$

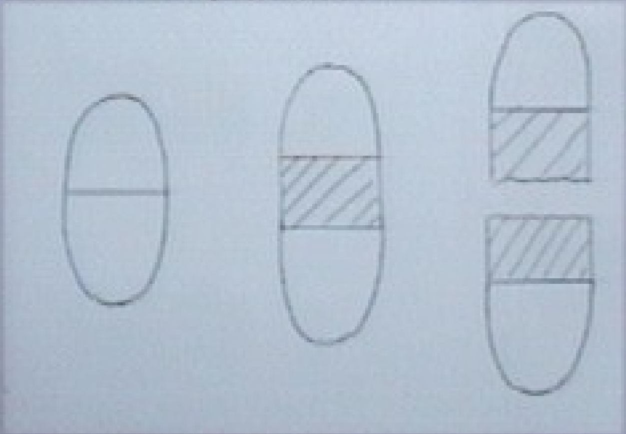
and the vector field $\mathbf{R} = J\frac{\partial}{\partial t}$ is *horizontal*, i.e., tangent to levels $t \times Y$, $t \in \mathbb{R}$.

Manifolds with cylindrical ends

A non-compact almost complex manifold (X, J) is said to have cylindrical ends if it is cylindrical outside a compact set.

Splitting

Given a real hypersurface Y in an almost complex manifold (X, J) one can split it into two manifolds with cylindrical (over Y ends) by a stretching of the neck procedure.



Any cylindrical structure on $\mathbb{R} \times Y$ is determined by the CR-structure ($\xi = JTY \cap TY, J_\xi = J|_\xi$) and the vector field $\mathbf{R} \in TY$ (which is transversal to ξ).

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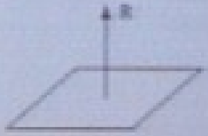

The distribution ξ and the vector field $\mathbf{R}$ uniquely determine a 1-form $\lambda = \lambda_J$ on Y which satisfies the conditions $\lambda(\mathbf{R}) \equiv 1$ and $\lambda|_\xi = 0$.

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Generically, all the periodic trajectories of the vector field $\mathbf{R}$ are non-degenerate, and thus there are finitely many of them of bounded period. We will call this case generic.
The set of simple periodic orbits will be denoted by $\mathcal{P}$.

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It is also useful to allow the, so-called Morse-Bott case when periodic orbits are organized in finite-dimensional manifolds, while the non-degeneracy condition for the flow of $\mathbf{R}$ is met in the transversal direction.

As in the case of general almost complex manifolds, to ensure the compactness results, it is necessary that J be compatible with certain symplectic data. We will assume that

A symplectic manifold (X, ω) is a manifold, locally modeled on $(\mathbb{R}^{2n}, dp \wedge dq)$. Equivalently, according to the Darboux theorem, (X, ω) is just a manifold with a non-degenerate closed differential 2-form.

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- there exists a closed 2-form ω of maximal rank on Y such that $i(\mathbf{R})\omega = 0$;
- the 1-form λ is preserved by the flow of $\mathbf{R}$, i.e. the Lie derivative $L_{\mathbf{R}}\lambda$ vanishes. This is equivalent to the condition $i(\mathbf{R})d\lambda = 0$.

The pair $\mathcal{H} = (\omega, \lambda)$ is called in this case (stable) Hamiltonian structure, and the cylindrical almost complex structure J is called compatible with $\mathcal{H}$.

There are 3 most important for us cases of Hamiltonian structures:

- The Floer case
- The contact case
- The circle bundle case

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$Y = M \times S^1$ is the mapping torus of a Hamiltonian symplectomorphism $f : M \to M$, generated by a 1-periodic time-dependant Hamiltonian H_t; $\mathbf{R} = \frac{\partial}{\partial t} + \mathrm{sgrad} H_t$, ξ is tangent bundle to the slices $M \times t$, $t \in S^1$. The periodic orbits of $\mathbf{R}$ are in 1-1 correspondence with the periodic points of $f : M \to M$.

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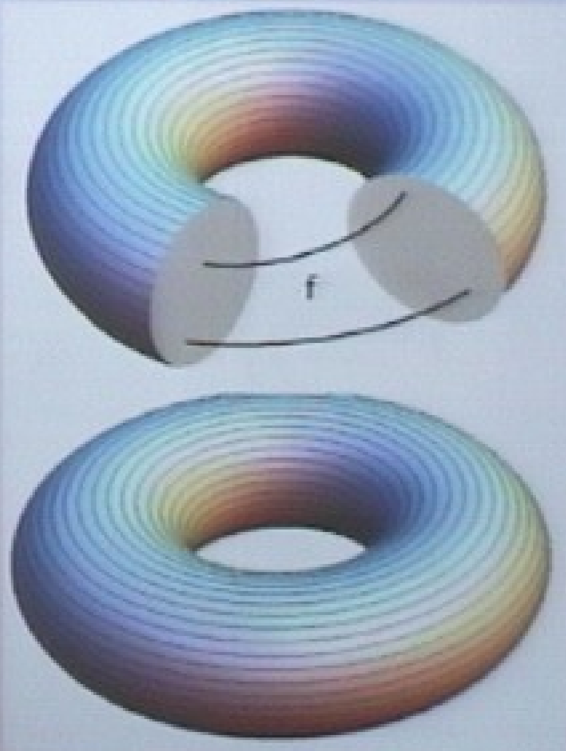

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$p : Y \to M$ is an S^1-bundle over a symplectic manifold (M, Ω). $\omega = p^*(\omega)$, λ is an S^1-connection form, $\mathbf{R}$ is the vector field which generates the S^1-action. In this case all orbits are periodic and thus $\mathcal{P} = M$.

Notice that for our choice of J the cylinder $\mathbb{R} \times \gamma \subset \mathbb{R} \times Y$ over a trajectory γ of the vector field $\mathbf{R}$ is always a J-holomorphic curve.

Category GEOM$_{SFT}$

- **OBJECTS:** Hamiltonian structures with compatible J
- **MORPHISMS:** Symplectic cobordisms with compatible J

$$\xrightarrow{SFT}$$

Category ALG$_{SFT}$

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- **Morphisms:** Symplectic cobordisms with compatible J

$\overset{SFT}{\Longrightarrow}$

Category Rational ALG$_{SFT}$

- **Objects:** Differential Poisson algebras
- **Morphisms:** Lagrangian correspondences

A *symplectic cobordism* between two Hamiltonian structures $\mathcal{H}_+ = (Y_+, \omega_+)$ and $\mathcal{H}_- = (H_-, \omega_-)$ is a symplectic manifold (X, ω) such that $\partial X = Y_+ \cup (-Y_-)$ and $\omega|_{Y_\pm} = \omega_\pm$. Note that "symplectic cobordism" is a partial order, and not an equivalence relation, because it is not symmetric. We will complete cobordisms by attaching cylindrical ends.

Assuming the non-degenerate case we call a periodic orbit γ of $\mathbf{R}$ even, or odd depending on the sign of the Leschetz number $\det(\mathrm{Id} - P_\gamma)$, where P_γ is the linearized Poincaré return map of the $\mathbf{R}$-flow along γ.

Let us associate with each **simple** orbit $\gamma \in P$ a sequence of graded variables $p_{k,\gamma}$, $q_{k,\gamma}$, $k \geq 1$.

They can be organized in a formal Fourier series

$$u_\gamma(x) = \sum_1^\infty p_{k,\gamma} e^{ikx} + q_{k,\gamma} e^{-ikx}.$$

A symplectic manifold (X, ω) is a manifold, locally modeled on $(\mathbb{R}^{2n}, dp \wedge dq)$. Equivalently, according to the Darboux theorem, (X, ω) is just a manifold with a non-degenerate closed differential 2-form.

An almost complex structure $J : TX \to TX$, $J^2 = -\mathrm{Id}$, is called compatible with ω if on each tangent space ω and J relate to each other as the standard symplectic form $dp \wedge dq$, and the standard complex structure do on $\mathbb{R}^{2n} = \mathbb{C}^n$. In other words, if

$$\omega(JV_1, V_2) - i\omega(V_1, V_2)$$

is a positive definite Hermitian form on TX.

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In a first approximation SFT associates with $(\mathcal{H}, J)$, where $\mathcal{H} = (\omega, \lambda)$ is a Hamiltonian structure and J a compatible almost complex structure, the following algebraic objects:

Objects in the Rational ($g = 0$) SFT

- A graded Poisson algebra P over C generated by $p_{k,\gamma}$, $q_{k,\gamma}$ with the following commutation relations: all elements Poisson commute (in the graded sense) except that $\{p_{k,\gamma}, q_{k,\gamma}\} = k$. We denote by S the corresponding symplectic space with coordinates $p_{k,\gamma}, q_{k,\gamma}$, and a symplectic form $\Omega = \sum_{k,\gamma} \frac{1}{k} dp_{k,\gamma} \wedge dq_{k,\gamma}$.

Objects in the Full SFT

Objects in the Rational ($g = 0$) SFT

- A graded Poisson algebra P over C generated by $p_{k,\gamma}$, $q_{k,\gamma}$ with the following commutation relations: all elements Poisson commute (in the graded sense) except that $\{p_{k,\gamma}, q_{k,\gamma}\} = k$. We denote by S the corresponding symplectic space with coordinates $p_{k,\gamma}, q_{k,\gamma}$, and a symplectic form $\Omega = \sum\limits_{k,\gamma} \frac{1}{k} dp_{k,\gamma} \wedge dq_{k,\gamma}$.
- An element $\mathbf{h} \in$ P, called Hamiltonian, which satisfies $\{\mathbf{h}, \mathbf{h}\} = 0$.

Objects in the Full SFT

- An associative Weyl algebra W over C generated by graded variables $p_{k,\gamma}$, $q_{k,\gamma}$ and an additional even graded element $\hbar$, with the following commutation relations: all elements commute (in the graded sense) except that $[p_{k,\gamma}, q_{k,\gamma}] = k\hbar$.
- An element $\mathbf{H} \in \frac{1}{\hbar}$W, called Hamiltonian, which satisfies $[\mathbf{H}, \mathbf{H}] = 0$.

The Weyl algebra $\frac{1}{\hbar}$W can be represented by differential operators acting on the "Fock space" Fock $= \{\sum g_m(q)\hbar^m\}$ on the left: $p_{k,\gamma} \mapsto \frac{\partial}{\partial q_{k,\gamma}}$.

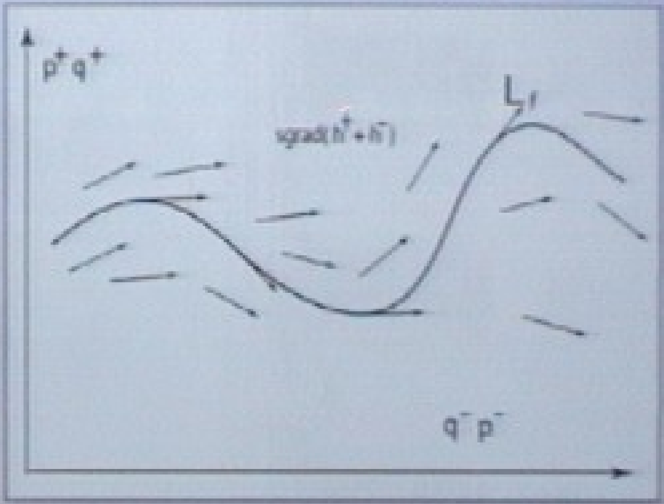
A morphism between $(\mathbf{P}^+, \mathbf{h}^+)$ and $(\mathbf{P}^-, \mathbf{h}^-)$ is a function $\mathbf{f}(q^-, p^+)$ such that

$$\mathbf{h}^+(q^+, p^+) + \mathbf{h}^-(q^-, p^-)\Big|_{L_{\mathbf{f}}} = 0, \tag{5}$$

where

$$L_{\mathbf{f}} = \left\{ q^+_{k,\gamma^+} = k\frac{\partial \mathbf{f}}{\partial p^+_{k,\gamma^+}};\ p^-_{k,\gamma^-} = k\frac{\partial \mathbf{f}}{\partial q^-_{k,\gamma^-}} \right\}.$$

The function $\mathbf{f}(q^-, p^+)$ generates a Lagrangian submanifold $L_{\mathbf{f}} \subset S^+ \times (-S^-)$, and (5) means that the Hamiltonian $\mathbf{h}^+(q^+, p^+) + \mathbf{h}^-(q^-, p^-)$ vanishes on $L_{\mathbf{f}}$.



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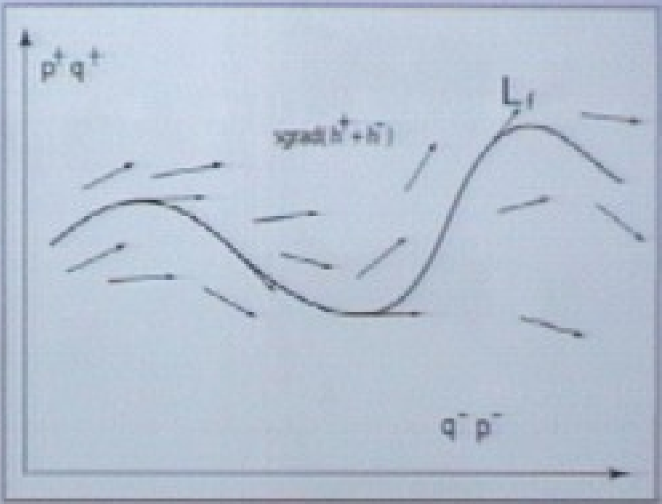

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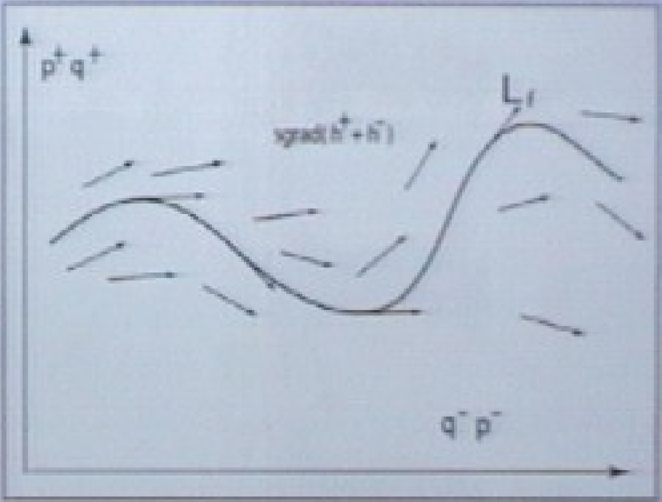

If $Y^- = \varnothing$ then L_f is a Lagrangian submanifold in $S = S^+$. Composition of morphisms is the composition of Lagrangian correspondences.

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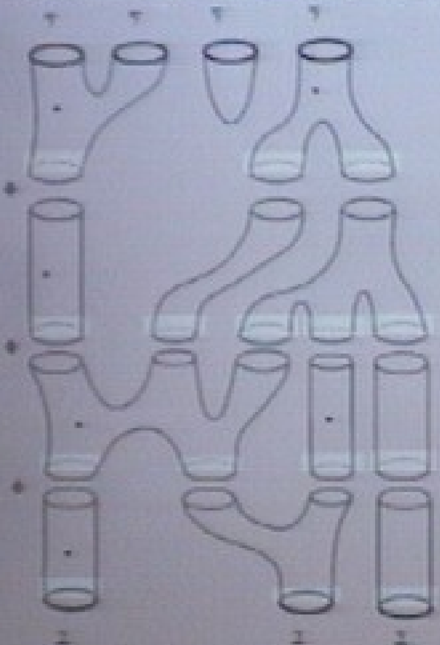

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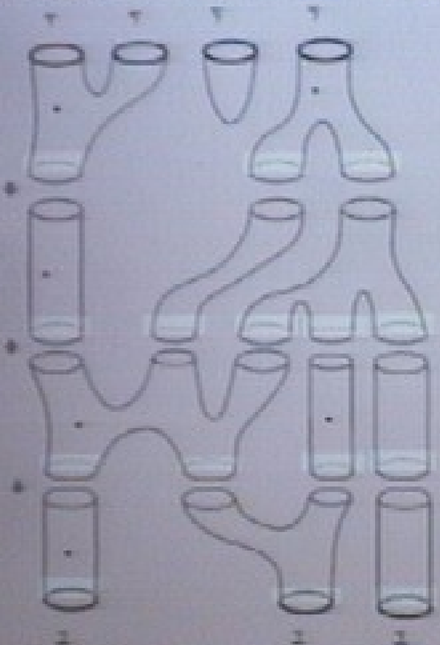


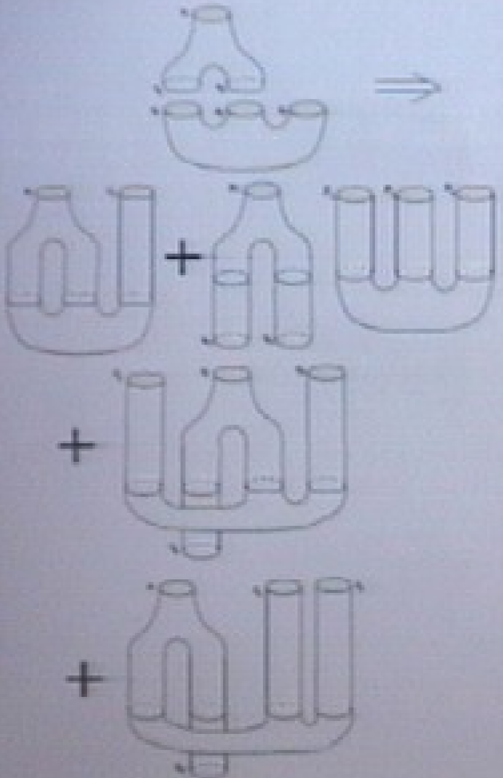
The codimension of a boundary stratum in the cylindrical case is equal to the number of components different from trivial cylinders. For manifolds with cylindrical ends it is equal to the number of non-trivial components in the cylindrical stories of the building. Thus in all cases the codimension 1 strata of the boundary consist only of 2-story buildings.

A symplectic manifold (X, ω) is a manifold, locally modeled on $(\mathbb{R}^{2n}, dp \wedge dq)$. Equivalently, according to the Darboux theorem, (X, ω) is just a manifold with a non-degenerate closed differential 2-form.

An almost complex structure $J : TX \to TX$, $J^2 = -\mathrm{Id}$, is called compatible with ω if on each tangent space ω and J relate to each other as the standard symplectic form $dp \wedge dq$, and the standard complex structure do on $\mathbb{R}^{2n} = \mathbb{C}^n$. In other words, if

$$\omega(JV_1, V_2) - i\omega(V_1, V_2)$$

is a positive definite Hermitian form on TX. Given ω, one can always find a compatible J (but not the other way around). Moreover, this choice is unique up to homotopy.



There are 4 different way to glue the two surfaces along their matching ends, i.e. the ends denoted by p's and q's with the same index. These 4 ways correspond to 4 terms in the composition formula for differential operators:

$(\hbar^{-1}p_1p_2p_3)\circ(\hbar^{-1}q_1q_2p_1) =$
$p_1p_3 + \hbar^{-2}q_1q_2p_1^2p_2p_3 +$
$\hbar^{-1}q_1p_1^2p_3 + \hbar^{-1}q_2p_1p_2p_3$.

An appearance of the coefficient k in the commutator $[p_{k,\gamma}, q_{k,\gamma}] = k\hbar$ (and the Poisson bracket $\{p_{k,\gamma}, q_{k,\gamma}\} = k$ in the rational case) corresponds to the fact that there are k distinct ways of gluing k-multiple orbits.

Given any set $\Theta = (\theta_1, \ldots, \theta_k)$ of (not necessarily closed) differential forms on Y, and sets Γ_-, Γ_+ of periodic orbits of $\mathbf{R}$ we define their correlator

$$\langle \theta_1, \ldots, \theta_k \rangle_{g,k}^{\Gamma_-,\Gamma_+} = \int\limits_{\overline{\mathcal{M}}_{g,k}^{\Gamma_-,\Gamma_+}(Y,J)/\mathbb{R}} \mathrm{ev}_1^*(\theta_1) \wedge \cdots \wedge \mathrm{ev}_k^*(\theta_k).$$

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then organize them into an element of $\frac{1}{\hbar}W$:

$$\langle \theta_1, \dots, \theta_k \rangle_k = \sum_{\Gamma_+,\Gamma_-,g} \langle \theta_1, \dots, \theta_k \rangle_{g,k}^{\Gamma_-,\Gamma_+} q^{\Gamma_+} p^{\Gamma_-} \hbar^{g-1}.$$

Given two sets of forms $\Theta = (\theta_1, \dots, \theta_k)$ and $\Upsilon = (\upsilon_1, \dots, \upsilon_m)$ we have $\mathbf{H}^{\Theta \cup \Upsilon}(\mathbf{t}, 0) = \mathbf{H}^{\Theta}(\mathbf{t})$, where $\mathbf{t} = (t_1, \dots, t_k)$. In particular, $\mathbf{H}^{\Theta}(0) = \mathbf{H}^{\varnothing} = \mathbf{H}$.

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"Master Equation":

$$\mathbf{H}^{\Theta \cup d\Theta} \circ \mathbf{H}^{\Theta \cup d\Theta} = D\mathbf{H}^{\Theta \cup d\Theta}, \quad D = \sum_1^k t_i \frac{\partial}{\partial s_i}, \qquad (6)$$

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In many interesting examples (e.g. in presence of an S^1-symmetry) $\mathbf{H}(0) = 0$. Hence, by differentiating (7) in t-variables we get $[\mathbf{H}_i, \mathbf{H}_j] = 0$, where $\mathbf{H}_i = \frac{\partial \mathbf{H}}{\partial t_i}(0)$, i.e. all $\mathbf{H}_j$ are commuting differential operators.

Symplectic Field Theory can be further upgraded to include descendent variables, in a similar vein, as it is done in the Gromov-Witten theory. In particular, $\mathbf{H}^{\Theta}$ becomes a function of infinitely many variables $T = (t_{ij}), i = 1, \ldots, k, j \geq 0$. Differentiating the identity $\mathbf{H}^{\Theta}(T) \circ \mathbf{H}^{\Theta}(T) = 0$, we observe existence of infinitely many commuting differential operators.

For instance, for $Y = S^1, \Theta = \{d\varphi\}$, one gets an infinite sequence of commuting integrals $\mathbf{H}_j, j = 0, \ldots$, for the quantized Burgers hierarchy

$$\frac{du}{dt_{k+2}} = \frac{1}{k!} u^k \frac{du}{dx}, \quad k = 0, 1, \ldots$$

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For $Y = S^3$ and $\Theta = \{$ harmonic volume form$\}$, one gets commuting integrals of the quantized dispersionless Toda hierarchy which includes the equation $\ddot{u} = -(e^{u-ix})_{xx}$, already appeared above in the problem of enumerating rational curves in CP^2.