Symplectic Field Theory (SFT) is a large, yet unfinished project which was initiated a few years ago by Alexander Givental, Helmut Hofer and the author.
A Symplectic manifold \((X, \omega)\) (unlike its symmetric counterpart: Riemannian manifold) has a huge infinite-dimensional group \(\mathcal{D} = \mathcal{D}(X, \omega)\) of symmetries, called symplectomorphisms.
Thanks to the “master equation” $[H, H] = 0$, we can define on $W$ a differential $D_H(A) = [A, H], A \in W$, which satisfies $D_H^2 = 0$. In many cases the homology $H_*(W, D_H)$ provides us with a powerful geometric invariant.
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Thanks to the "master equation" $[\mathbf{H}, \mathbf{H}] = 0$, we can define on $W$ a differential $D_{\mathbf{H}}(A) = [A, \mathbf{H}]$, $A \in W$, which satisfies $D_{\mathbf{H}}^2 = 0$. In many cases the homology $H_*(W, D_{\mathbf{H}})$ provides us with a powerful geometric invariant.

- In the Floer case, this gives a far-going generalization of the Floer homology theory, by bringing to it new invariant algebraic structures.
- In the contact case this leads to the contact homology theory, which, for contact manifolds of dimension $> 3$, is essentially the only known source of invariants.

Alternatively, one may use $\mathbf{H}$ to define a differential $d_{\mathbf{H}}$ on the Fock space $\text{Fock} = \{ \mathbf{g} = \sum_{k \geq 0} g_k(q) h^k \}$:

$$d_{\mathbf{H}}(\mathbf{g}) = \mathbf{H} \mathbf{g},$$

which leads to a structure of a $\text{BV}_\infty$-algebra on $\text{Fock}$. This formalism was recently explored by Cieliebak-Latchev.
Let \((X, \omega)\) be a symplectic cobordism which bounds (on its positive end) a Hamiltonian structure \(\mathcal{H} = (\omega, \lambda)\) on \(Y = \partial X\), and \(J\) a compatible almost complex structure on \(Y\).

As it was explained above, the "naive" Rational SFT associates with \((X, \omega, J)\) a Lagrangian submanifold \(L_f \subset S\) generated by a function \(f(p)\), such that the Hamiltonian \(h(q, p)\) vanishes on \(L_f\).
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In a little bit more advanced version of Rational SFT, in a way similar to how this was done above for the Hamiltonian, we associate with a sequence \(\Theta = (\theta_1, \ldots, \theta_k)\) of differential forms on \(X\) a function

\[
f^\Theta(T, q, p) \in P \otimes \mathbb{C}[T],
\]

where \(\mathbb{C}[T]\) is a graded polynomial algebra generated by graded variables \(T = (t_{ij}^j), i = 1, \ldots, k; j \geq 0\).

If \(d\Theta = 0\) then similarly to the "naive case", \(h^{\Theta|_Y}\) vanishes on \(L_{f\Theta}\).
Take now $\Theta$ with $d\Theta|_Y = 0$. Then we have

\[ h^{\Theta_Y}(T)|_{L_{\Theta\cup d\Theta}(T,S)} = D F^{\Theta\cup d\Theta}(T,S), \] (8)

where $D = \sum_{i,j} t_{ij} \frac{\partial}{\partial s_{ij}}$ and $S = (s_{ij})$ are variables associated to $d\Theta = (d\theta_1, \ldots, d\theta_k)$.

Take, for instance, $Y = S^3$, $X$ is a 4-ball, $\Theta = (\theta)$ where $\theta$ is a 3-form which restricts to the standard volume form on $S^3$. Then, by differentiating (8) with respect to $t = t_{10}$ and setting $T = 0$ (and denoting $f = f^{\Theta}$, $s = s_{10}$), we get
Take now $\Theta$ with $d\Theta|_Y = 0$. Then we have
\[
h^{\Theta Y}(T)|_{L_{t_{10}\phi_0(T,S)}} = D\Phi^{\Theta Y}(T,S),
\]
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Take, for instance, $Y = S^3$, $X$ is a 4-ball, $\Theta = (\theta)$ where $\theta$ is a 3-form which restricts to the standard volume form on $S^3$. Then, by differentiating (8) with respect to $t = t_{10}$ and setting $T = 0$ (and denoting $f = f^{(r)}$, $s = s_{10}$), we get
\[
\frac{\partial f}{\partial s}(S) = \frac{\partial h}{\partial t}(0)|_{L_{t_{10}(S)}},
\]

which is the Hamilton-Jacobi equation for the evolution of the Lagrangian manifold $L_s$, $s = s_{10}$, under the flow of the Hamiltonian $h_1 = \frac{\partial h^r(T)}{\partial t}(0)$. 
Applications to Hamiltonian Dynamics
Final Remarks: To Do List

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- Relations with the Theory of (Quantum) Integrable Systems
- Algebra of SFT: what is invariant?
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A Symplectic manifold \((X, \omega)\) (unlike its symmetric counterpart: Riemannian manifold) has a huge infinite-dimensional group \(\mathcal{D} = \mathcal{D}(X, \omega)\) of symmetries, called symplectomorphisms. For instance, for \(n = 1\) a symplectic form is just an area form, and thus symplectomorphisms are area preserving transformations. However, for \(n > 1\), the group of symplectomorphisms is a proper \(C^0\)-closed subgroup of the group of volume preserving transformations.
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The Lie algebra \(\mathfrak{d}\) of \(\mathfrak{D}\) consists of symplectic vector fields, i.e. tangent vector fields \(\nu \in TM\) such that the form \(\alpha = i(\nu)\omega\) is closed. If \(\alpha\) is exact, \(\alpha = dH\), then the vector field \(\nu = \text{sgrad}H\) is called Hamiltonian, and the function \(H\) its Hamiltonian function.
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Lagrangian, i.e. half-dimensional isotropic submanifolds $L \subset X$, $\omega|_L = 0$, play a very important role in symplectic geometry. Given a compatible $J$, $L$ is Lagrangian $\iff JTL \perp L$. Here are two important examples of Lagrangian submanifolds:
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Here are two important examples of Lagrangian submanifolds:

- A section $s : M \to T^*M$ is Lagrangian if and only if it is a closed 1-form.

- Given a symplectic manifold $(X, \omega)$ a map $f : X \to X$ is symplectic, i.e. $f^*\omega = \omega$, if and only if its graph $\Gamma_f \subset X \times X$ is Lagrangian with respect to the symplectic form $\Omega = \omega \times (-\omega)$ on $X \times X$. 
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- the total volume (i.e. $\int_X \omega^n$);
- the homotopy class of a compatible almost complex structure $J$, and
- the cohomology class $[\omega] \in H^2(X)$, in the case of a closed manifold $X$.

It was great Gromov's insight, when he introduced holomorphic curves as a tool for finding more subtle, specifically symplectic invariants.
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Given two almost complex manifolds \((S, J_S)\) and \((X, J_X)\) it makes sense to talk about holomorphic maps, i.e. maps \(f : S \to X\) which satisfy

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- If \(\dim S > 2\) the system (1) is overdetermined, and usually has no solutions, even locally.
- When \(\dim S = 2\), i.e. when \((S, J_S)\) is a Riemann surface then the system (1) is determined, and regardless of integrability of \(J_X\), the local theory of holomorphic maps \(S \rightarrow X\), or as they called in this case, holomorphic (also \(J\)-holomorphic, pseudo-holomorphic) curves, is as rich as in the integrable case.
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Thus, under certain transversality assumptions the moduli spaces of holomorphic curves form finite dimensional manifolds, or at least orbifolds.
Gromov compactness theorem for holomorphic curves provides a compactification of the moduli spaces of holomorphic curves, similar to the Knudsen-Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n$ marked points.
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$$(S_g, j, x_1, \ldots, x_n)$$

of conformal structures on a closed surface $g$ with $n$ marked points, can be identified for $2g + n \geq 3$, with the moduli space of complete hyperbolic metrics of finite area on $S_g \setminus \{x_1, \ldots, x_n\}$. 
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The compactification of the moduli space of holomorphic curves provided by Gromov compactness theorem (in Kontsevich's interpretation) differs only in the stability condition: it needs to be satisfied only for constant components of nodal curves (which are sometimes called ghosts).

In particular one may have a phenomenon of bubbling off of holomorphic spheres.
Symplectic Field Theory (SFT) is a large, yet unfinished project which was initiated a few years ago by Alexander Givental, Helmut Hofer and the author. Several other mathematicians contributed and keep contributing a lot of work towards the foundations and applications of SFT: C. Abbas, F. Bourgeois, K. Cieliebak, T. Coates, T. Ekholm, J. Etnyre, R. Hind, A. Ivrii, D. McDuff, E. Katz, S.-S. Kim, J. Latchev, L. Ng, K. Mohnke, K. Wysocki, B. Parker, R. Siefring, J. Sabloff, M. Sullivan, M.-L. Yau, I. Ustićovsky, E. Zehnder,…..

We also benefited a lot from discussions and collaboration with the people working on tightly related subjects, (e.g. relative Gromov-Witten theory, Cluster Homology, String Topology etc.).
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The role of a compatible with $J$ symplectic form $\omega$ is to ensure the area bounds for $J$-holomorphic curves. Indeed, the area of any holomorphic curve $f : S \to X$ coincides with its symplectic area $\int f^* \omega$, and hence for a closed $S$ the area of a holomorphic curve $f : S \to X$ depends only on its homology class in $H_2(X)$. In particular one may have a phenomenon of bubbling off of holomorphic spheres.
Gromov’s scheme of defining symplectic invariants via the holomorphic curve theory was the following:

- Pick a (generic) almost complex structure $J$ compatible with $\omega$.
- Then, the compactified moduli space $\overline{M}_g^A(X, J)$ of $J$-holomorphic curves of genus $g$ in a fixed homology class $A \in H_2(X)$ form a cycle in the similarly enriched moduli space $\overline{\mathcal{F}}_g^A(X)$ of all smooth maps $F : (S_g, j) \to X$. 
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- The compactness theorem ensures that the homology class of this cycle remains unchanged when one varies $J$, while keeping it compatible with $\omega$.

- But all $J$, which are compatible with a fixed $\omega$, are homotopic, and hence the homology class of $\overline{M}_g^A(X,J)$ in $\overline{F}_g^A(X)$, called Gromov invariant, depends only on $\omega$. 
Theorem (Gromov)

Take $\mathbb{R}^3 = \{p_2 = 0\} \subset \mathbb{R}^4$. Let $B$ be the unit ball
$\{p_1^2 + q_1^2 + q_2^2 < 1\} \subset \mathbb{R}^3$ and set
$H = \mathbb{R}^3 \setminus B$. Let $h_\pm : B^4(R) \to \mathbb{R}^4_\pm$ be two orthogonal embeddings of the
4-ball of radius $R > 1$ into the upper- and lower-half spaces
$\mathbb{R}^4_+ = \{p_2 > 0\}$ and $\mathbb{R}^4_- = \{p_2 < 0\}$.
Then $h_-$ and $h_+$ are not symplectically isotopic in $\mathbb{R}^4 \setminus H$. 

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$\{p_1^2 + q_1^2 + q_2^2 < 1\} \subset \mathbb{R}^3$ and set $H = \mathbb{R}^3 \setminus B$. Let $h_\pm : B^4(R) \to \mathbb{R}_\pm^4$ be two orthogonal embeddings of the 4-ball of radius $R > 1$ into the upper- and lower-half spaces $\mathbb{R}_+^4 = \{p_2 > 0\}$ and $\mathbb{R}_-^4 = \{p_2 < 0\}$. Then $h_-$ and $h_+$ are not symplectically isotopic in $\mathbb{R}_-^4 \setminus H$. 
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Yakov Eliashberg  Stanford University
Symplectic Field Theory and its Applications
Suppose there exists a symplectic isotopy \( h_t : B^4(R) \to \mathbb{R}^4 \setminus H \) connecting \( h_0 = h_+ \) and \( h_1 = h_- \). For each \( t \in [0, 1] \), choose an almost complex structure \( J_t \) on \( \mathbb{R}^4 \), such that

- \( J_0 = J_1 \) is the standard structure on \( \mathbb{R}^4 = \mathbb{C}^2 \);
- \( J_t = J_0 \) at infinity and near \( L \) for all \( t \in [0, 1] \);
- \( J_t \) is compatible with \( \omega = dp \wedge dq \);
- \( J_t |_{h_t(B^4(R))} = (h_t)^* J_0 \).
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Take a Lagrangian cylinder $L = \{p_1^2 + q_1^2 = 1; \ p_2 = 0\} \subset \mathbb{R}^3 \subset \mathbb{R}^4$. 

\[ p_2 = 0. \quad B^3(1) \]
Suppose there exists a symplectic isotopy $h_t : B^4(R) \to \mathbb{R}^4 \setminus H$ connecting $h_0 = h_+$ and $h_1 = h_-$. For each $t \in [0, 1]$, choose an almost complex structure $J_t$ on $\mathbb{R}^4$, such that:

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Take a Lagrangian cylinder $L = \{ p_1^2 + q_1^2 = 1; \ p_2 = 0 \} \subset \mathbb{R}^3 \subset \mathbb{R}^4$. Let $\mathcal{M}_t$ be the moduli space of $J_t$-holomorphic discs with 1 marked point in $\mathbb{R}^4$ with boundary in $L$. 
The evaluation map $\text{ev} : \mathcal{M}_t \to \mathbb{R}^4$ provides a smooth hyperface \( \Sigma_t \) bounded by \( L \), and center of the moving ball \( h_t(B^4(R)) \) should hit \( \Sigma_t \) at a certain moment \( t = t_0 \).
Symplectic Geometry is the geometry of a non-degenerate skew-symmetric bilinear form.
The evaluation map $\text{ev} : M_t \to \mathbb{R}^+$ provides a smooth hypersurface $\Sigma_t$ bounded by $L$, and center of the moving ball $h_t(B^4(R))$ should hit $\Sigma_t$ at a certain moment $t = t_0$. 
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Area $\left( h_{t_0}^{-1}(D_{t_0}) \right)$ 

$$ = \int_{h_{t_0}(B^4(R)) \cap D_{t_0}} \omega$$

$$ \leq \int_{D_{t_0}} \omega = \pi.$$
The evaluation map $ev: M_t \to \mathbb{R}^d$ provides a smooth hyperface $\Sigma_t$ bounded by $L$, and center of the moving ball $h_t(B^4(R))$ should hit $\Sigma_t$ at a certain moment $t = t_0$.

\[
\text{Area} \left( h_{t_0}^{-1}(D_{t_0}) \right) = \int_{h_{t_0}(B^4(R)) \cap D_{t_0}} \omega \leq \int_{D_{t_0}} \omega = \pi.
\]

On the other hand,

\[
\text{Area} \left( h_{t_0}^{-1}(D_{t_0}) \right) \geq \pi R^2.
\]
Gromov-Witten potential is a certain, motivated by Physics way to package algebraic information contained in the Gromov invariant.
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\[
(\theta_1, \ldots, \theta_k)_{g, k}^A = \int_{\mathcal{M}_{g, k}^A(X, J)} \prod_{i=1}^k \text{ev}_i^*(\theta_i), \quad (2)
\]

where
Gromov-Witten potential is a certain, motivated by Physics way to package algebraic information contained in the Gromov invariant. Let \((X, \omega, J)\) be a symplectic manifold with a compatible almost complex structure \(J\). Given cohomology classes \(\theta_1, \ldots, \theta_k \in H^*(X)\) we define their correlator

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\langle \theta_1, \ldots, \theta_k \rangle_{g,k}^A = \int_{\mathcal{M}_{g,k}^A(X, J)} \ev_1^*(\theta_1) \wedge \cdots \wedge \ev_k^*(\theta_k),
\]

(2)

where

- \(\mathcal{M}_{g,k}^A(X, J)\) is the compactified moduli space of \(J\)-holomorphic curves of genus \(g\) with \(k\) marked points \(x_1, \ldots, x_k \in S_g\) in a given homology class \(A \in H_2(X)\);
- \(\ev_i : \mathcal{M}_{g,n}^A(X, J) \to X\) is the evaluation map at the point \(x_i\), \(i = 1, \ldots, k\).
Let us fix a basis \( \theta_1, \ldots, \theta_N \in H^*(X) \) and a basis \( A_1, \ldots, A_K \in H^2(X) \). Let \( (t_1, \ldots, t_N) \) be the graded coordinates in \( H^*(X) \) in the chosen basis. We will also identify a homology class \( A \in H_2(X) \) with its degree \( d = (d_1, \ldots, d_K) \), i.e. coordinates in the basis \( A_1, \ldots, A_m \). Let us also introduce (even graded) variables \( z_1, \ldots, z_K \) and \( \hbar \), and write \( z^d = z_1^{d_1} \cdots z_K^{d_K} \).
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Then the Gromov-Witten potential $F(t_0, \ldots, t_N, z)$ is defined by the formula

$$F(t_0, \ldots, t_N, z) = \sum_{g,d,k} \frac{1}{k!} \langle t, \ldots, t \rangle_{g,k}^d z^d h^{g-1},$$

where $t = t_1 \theta_1 + \cdots + t_N \theta_N$. 

(3)
Let us fix a basis $\theta_1, \ldots, \theta_N \in H^*(X)$ and a basis $A_1, \ldots, A_K \in H^2(X)$. Let $(t_1, \ldots, t_N)$ be the graded coordinates in $H^*(X)$ in the chosen basis. We will also identify a homology class $A \in H_2(X)$ with its degree $d = (d_1, \ldots, d_K)$, i.e. coordinates in the basis $A_1, \ldots, A_m$. Let us also introduce (even graded) variables $z_1, \ldots, z_K$ and $\hbar$, and write $z^d = z_1^{d_1} \ldots z_K^{d_K}$.

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$$F(t_0, \ldots, t_N, z) = \sum_{g, d, k} \frac{1}{k!} (t_0, \ldots, t_N)^d_{g, k} z^d \hbar^{g-1}.$$  \hspace{1cm} (3)

where $t = t_1 \theta_1 + \cdots + t_N \theta_N$. 
One can further upgrade the Gromov-Witten potential to the so-called descendent potential and make it dependent on infinitely many variables $T = (t_{ij})$, $i = 1, \ldots, k$, $j = 0, 1, \ldots$:

$$F(T, z) = \sum_{g,d,k} \frac{1}{k!} \langle T, \ldots, T \rangle_{g,k}^d z^d \eta^{g-1}.$$ (4)

where:

$$\langle T, \ldots, T \rangle_{g,k}^A = \int_{\overline{M}_{g,k}^A(X, J)} \wedge^k \left( \sum_{i \in \{1, \ldots, k\}, j \geq 0} t_{ij} \text{ev}_i^*(\theta_i)(c_1(L_m))^j \right).$$

where $L_m$ is the tautological line bundle over $\overline{M}_{g,m}(X, J)$ formed by cotangent lines at the marked point $x_m$. 
Symplectic Geometry is the geometry of a non-degenerate skew-symmetric bilinear form. Such form exists only on an even-dimensional space, and is unique up to an isomorphism. Viewed as a differential form, it can be written in $\mathbb{R}^{2n} = \{p_1, q_1, \ldots, p_n, q_n\}$ as

$$\sum_{j=1}^{n} dp_j \wedge dq_j = dp \wedge dq.$$
The genus 0 part $F^{(0)}(t_0, \ldots, t_N, z)$ of $F = \sum_{g=0}^{\infty} F^{(g)} t^g z^{g-1}$, is called the rational Gromov-Witten potential.

For instance, for $X = \mathbb{C}P^2$ let us choose the standard basis $\theta_0 = 1, \theta_1 \in H^2(\mathbb{C}P^2), \theta_2 \in H^4(\mathbb{C}P^2)$ and choose $[\mathbb{C}P_1]$ as a generator of $H_2(\mathbb{C}P_2)$. Consider the function

$$f(t, z) = F^{(0)}(0, 0, t, z) = \sum_{d=1}^{\infty} \sum_{k=1}^{\infty} N_{d, k} t^k z^d.$$
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The coefficients $N_{d,k}$ have a simple enumerative meaning: these are the numbers of rational curves of a given degree $d$ in the complex projective plane, which pass through $k$ points in general position.
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$$f(t, z) = F^{(0)}(0, 0, t, z) = \sum_{d=1}^{\infty} \sum_{k=1}^{\infty} N_{d,k} t^k z^d.$$  

The coefficients $N_{d,k}$ have a simple enumerative meaning: these are the numbers of rational curves of a given degree $d$ in the complex projective plane, which pass through $k$ points in general position.

$N_{1,2} = 1, N_{2,5} = 1, N_{3,8} = 12, N_{4,11} = 620, \ldots$ In general, $k = 3d - 1$ if $N_{d,k} \neq 0$. 
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For instance, for \( X = \mathbb{C}P^2 \) let us choose the standard basis \( \theta_0 = 1, \theta_1 \in H^2(\mathbb{C}P^2), \theta_2 \in H^4(\mathbb{C}P^2) \) and choose \([\mathbb{C}P_1]\) as a generator of \( H_2(\mathbb{C}P^2)\). Consider the function

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We will describe here a more geometric recipe for computing the function \( f(t, z) \), which is provided by Symplectic Field Theory.
Consider the space $S$ of $\mathbb{C}^2$-valued formal Fourier series

$$U(x) = (u_0(x), u_1(x)) = \sum_{m=1}^{\infty} P_m e^{imx} + Q_m e^{-imx},$$

where $P_m = (p_{m,0}, p_{m,1})$, $Q_m = (q_{m,0}, q_{m,1})$. 
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where $P_m = (p_{m,0}, p_{m,1})$, $Q_m = (q_{m,0}, q_{m,1})$. We endow $S$ with a symplectic form $\Omega = \sum \frac{1}{m} (dp_{m,0} \wedge dq_{m,1} + dp_{m,1} \wedge dq_{m,0})$. It corresponds to a Poisson tensor

$$P(\delta U, \delta V) = \frac{1}{2\pi i} \int_0^{2\pi} \langle \delta U, \delta V' \rangle \, dx,$$

where $\langle \cdot, \cdot \rangle$ is the inner product with the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, i.e.

$$\langle A, B \rangle = a_0 b_1 + a_1 b_0$$

for $A = (a_0, a_1)$, $B = (b_0, b_1) \in \mathbb{C}^2$. 
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\( (A, B) = a_0 b_1 + a_1 b_0 \) for $A = (a_0, a_1)$, $B = (b_0, b_1) \in \mathbb{C}^2$. 
Consider a Hamiltonian function $H$ on $S$ defined by the formula

$$H(U) = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{u_0^2(x)}{2} + e^{u_1(x)-ix} \right) dx,$$

where $U = (u_0, u_1)$.

The flow $\Phi^t$ of this Hamiltonian is given by differential equations

$$\dot{u}_0 = -i \frac{d}{dx} \left( e^{u_1(x)-ix} \right);$$

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Note that this is a dispersionless Toda equation which can be rewritten as

$$\ddot{u}_1 = - \left( e^{u_1-ix} \right)_{xx}.$$

(I learned this from B. Dubrovin.)
Denote by $L_t$ the image $\Phi^t(L)$ of the Lagrangian subspace $L = \{ Q = 0 \}$ under the Hamiltonian flow $\Phi^t$. If $L_t$ is graphical with respect to the projection $(P, Q) \mapsto P$ (and in the world of formal power series everything is graphical!) then it is defined by a generating function $G_t(P)$,

$$L_t = \begin{cases} 
q_{m,0} = m \frac{\partial G_t}{\partial p_{m,0}}; \\
q_{m,1} = m \frac{\partial G_t}{\partial p_{m,1}}.
\end{cases}$$
Symplectic Geometry is the geometry of a non-degenerate skew-symmetric bilinear form. Such form exists only on an even-dimensional space, and is unique up to an isomorphism. Viewed as a differential form, it can be written in $\mathbb{R}^{2n} = \{p_1, q_1, \ldots, p_n, q_n\}$ as

$$\sum_{j=1}^{n} dp_j \wedge dq_j = dp \wedge dq.$$ 

If one identifies $\mathbb{R}^{2n} = \{p_1, q_1, \ldots, p_n, q_n\}$ with $\mathbb{C}^n = \{z_1 = p_1 + iq_1, \ldots, p_n + iq_n\}$, then for any two vectors $V_1, V_2 \in \mathbb{R}^{2n}$ we have $\omega_0(V_1, V_2) = \langle V_1, iV_2 \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{2n}$. Equivalently, $-\omega$ is the imaginary part of the standard Hermitian product on $\mathbb{C}^n$. 
The Gromov-Witten potential $F$ is a symplectic invariant, i.e. it depends on $\omega$, and not $J$ and other additional choices which we made above.
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Cylindrical almost complex manifolds.

An almost complex structure $J$ on $\mathbb{R} \times Y$ is called \textit{cylindrical} if it is invariant under translations

$$(t, x) \mapsto (t + c, y), \; t, c \in \mathbb{R}, \; y \in Y,$$

and the vector field $R = J \frac{\partial}{\partial t}$ is \textit{horizontal}, i.e., tangent to levels $t \times Y, \; t \in \mathbb{R}$.

Manifolds with cylindrical ends

A non-compact almost complex manifold $(X, J)$ is said to have \textit{cylindrical ends} if it is cylindrical outside a compact set.
Splitting

Given a real hypersurface $Y$ in an almost complex manifold $(X, J)$ one can split it into two manifolds with cylindrical (over $Y$ ends) by a stretching of the neck procedure.
Any cylindrical structure on $\mathbb{R} \times Y$ is determined by the CR-structure ($\xi = JTY \cap TY$, $J_\xi = J_{\xi|\xi}$) and the vector field $R \in TY$ (which is transversal to $\xi$).
Any cylindrical structure on $\mathbb{R} \times Y$ is determined by the CR-structure $(\xi = J^TY \cap TY, J|_\xi = J|_\xi)$ and the vector field $R \in TY$ (which is transversal to $\xi$).

The distribution $\xi$ and the vector field $R$ uniquely determine a 1-form $\lambda = \lambda_J$ on $Y$ which satisfies the conditions $\lambda(R) \equiv 1$ and $\lambda|_\xi = 0$. 
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Generically, all the periodic trajectories of the vector field $R$ are non-degenerate, and thus there are finitely many of them of bounded period. We will call this case generic. The set of simple periodic orbits will be denoted by $\mathcal{P}$.
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Generically, all the periodic trajectories of the vector field $R$ are non-degenerate, and thus there are finitely many of them of bounded period. We will call this case generic.

The set of simple periodic orbits will be denoted by $\mathcal{P}$. It is also useful to allow the, so-called Morse-Bott case when periodic orbits are organized in finite-dimensional manifolds, while the non-degeneracy condition for the flow of $R$ is met in the transversal direction.
As in the case of general almost complex manifolds, to ensure the compactness results, it is necessary that $J$ be compatible with certain symplectic data. We will assume that
A symplectic manifold \((X, \omega)\) is a manifold, locally modeled on \((\mathbb{R}^{2n}, dp \wedge dq)\). Equivalently, according to the Darboux theorem, 
\((X, \omega)\) is just a manifold with a non-degenerate closed differential 2-form.
As in the case of general almost complex manifolds, to ensure the compactness results, it is necessary that $J$ be compatible with certain symplectic data. We will assume that

- there exists a closed 2-form $\omega$ of maximal rank on $Y$ such that $i(R)\omega = 0$;
- the 1-form $\lambda$ is preserved by the flow of $R$, i.e. the Lie derivative $L_R \lambda$ vanishes. This is equivalent to the condition $i(R)d\lambda = 0$.

The pair $\mathcal{H} = (\omega, \lambda)$ is called in this case (stable) Hamiltonian structure, and the cylindrical almost complex structure $J$ is called compatible with $\mathcal{H}$. 
There are 3 most important for us cases of Hamiltonian structures:

- The Floer case
- The contact case
- The circle bundle case
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- **The circle bundle case**

\[ Y = M \times S^1 \] is the mapping torus of a Hamiltonian symplectomorphism \( f : M \rightarrow M \), generated by a 1-periodic time-dependant Hamiltonian \( H_t; R = \frac{\partial}{\partial t} + \text{sgrad} H_t \). \( \xi \) is tangent bundle to the slices \( M \times t, t \in S^1 \). The periodic orbits of \( R \) are in 1-1 correspondence with the periodic points of \( f : M \rightarrow M \).
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\[ p : Y \to M \] is an \( S^1 \)-bundle over a symplectic manifold \( (M, \Omega) \), \( \omega = p^*(\omega) \), \( \lambda \) is an \( S^1 \)-connection form, \( R \) is the vector field which generates the \( S^1 \)-action. In this case all orbits are periodic and thus \( \mathcal{P} = M \).
Notice that for our choice of $J$ the cylinder $\mathbb{R} \times \gamma \subset \mathbb{R} \times Y$ over a trajectory $\gamma$ of the vector field $R$ is always a $J$-holomorphic curve.
Category $\text{GEOM}_{SFT}$

- **Objects:** Hamiltonian structures with compatible $J$
- **Morphisms:** Symplectic cobordisms with compatible $J$

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- **Objects:** Differential Weyl algebras.
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A *symplectic cobordism* between two Hamiltonian structures $\mathcal{H}_+ = (Y_+, \omega_+)$ and $\mathcal{H}_- = (H_-, \omega_-)$ is a symplectic manifold $(X, \omega)$ such that $\partial X = Y_+ \cup (-Y_-)$ and $\omega|_{V_\pm} = \omega_\pm$. Note that "symplectic cobordism" is a partial order, and not an equivalence relation, because it is not symmetric. We will complete cobordisms by attaching cylindrical ends.
Assuming the non-degenerate case we call a periodic orbit $\gamma$ of $\mathbb{R}$ even, or odd depending on the sign of the Leschetz number $\det(\text{Id} - P_\gamma)$, where $P_\gamma$ is the linearized Poincaré return map of the $\mathbb{R}$-flow along $\gamma$.

Let us associate with each simple orbit $\gamma \in \mathcal{P}$ a sequence of graded variables $p_{k,\gamma}, q_{k,\gamma}, k \geq 1$.

They can be organized in a formal Fourier series

$$u_\gamma(x) = \sum_{k=1}^{\infty} p_{k,\gamma} e^{ikx} + q_{k,\gamma} e^{-ikx}.$$
A symplectic manifold \((X, \omega)\) is a manifold, locally modeled on \((\mathbb{R}^{2n}, dp \wedge dq)\). Equivalently, according to the Darboux theorem, \((X, \omega)\) is just a manifold with a non-degenerate closed differential 2-form.

An almost complex structure \(J: TX \to TX, J^2 = -\text{Id}\), is called compatible with \(\omega\) if on each tangent space \(\omega\) and \(J\) relate to each other as the standard symplectic form \(dp \wedge dq\), and the standard complex structure do on \(\mathbb{R}^{2n} = \mathbb{C}^n\). In other words, if

\[
\omega(JV_1, V_2) - i\omega(V_1 \wedge V_2)
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is a positive definite Hermitian form on \(TX\).
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In a first approximation SFT associates with $(\mathcal{H}, J)$, where $\mathcal{H} = (\omega, \lambda)$ is a Hamiltonian structure and $J$ a compatible almost complex structure, the following algebraic objects:
A graded Poisson algebra $P$ over $\mathbb{C}$ generated by $p_{k,\gamma}$, $q_{k,\gamma}$ with the following commutation relations: all elements Poisson commute (in the graded sense) except that $\{p_{k,\gamma}, q_{k,\gamma}\} = k$. We denote by $S$ the corresponding symplectic space with coordinates $p_{k,\gamma}$, $q_{k,\gamma}$, and a symplectic form $\Omega = \sum_k \frac{1}{k} dp_{k,\gamma} \wedge dq_{k,\gamma}$.
Objects in the Rational ($g = 0$) SFT

- A graded Poisson algebra $P$ over $\mathbb{C}$ generated by $p_{k,\gamma}$, $q_{k,\gamma}$ with the following commutation relations: all elements Poisson commute (in the graded sense) except that $\{p_{k,\gamma}, q_{k,\gamma}\} = k$. We denote by $S$ the corresponding symplectic space with coordinates $p_{k,\gamma}$, $q_{k,\gamma}$, and a symplectic form $\Omega = \sum_{k,\gamma} \frac{1}{k} dp_{k,\gamma} \wedge dq_{k,\gamma}$.

- An element $h \in P$, called Hamiltonian, which satisfies $\{h, h\} = 0$.

Objects in the Full SFT

- An associative Weyl algebra $W$ over $\mathbb{C}$ generated by graded variables $p_{k,\gamma}$, $q_{k,\gamma}$ and an additional even graded element $h$, with the following commutation relations: all elements commute (in the graded sense) except that $[p_{k,\gamma}, q_{k,\gamma}] = kh$.

- An element $H \in \frac{1}{h} W$, called Hamiltonian, which satisfies $[H, H] = 0$.

The Weyl algebra $\frac{1}{h} W$ can be represented by differential operators acting on the "Fock space" $\text{Fock} = (\sum g_m(q) h^m)$ on the left: $p_{k,\gamma} \mapsto \frac{\partial}{\partial q_{k,\gamma}}$. 
A morphism between \((P^+, h^+)\) and \((P^-, h^-)\) is a function 
\(f(q^-, p^+)\) such that

\[
h^+(q^+, p^+) + h^-(q^-, p^-) \bigg|_{L_f} = 0, \tag{5}
\]

where

\[
L_f = \left\{ q^+_{k, \gamma^+} = k \frac{\partial f}{\partial p^+_{k, \gamma^+}}, \quad p^-_{k, \gamma^-} = k \frac{\partial f}{\partial q^-_{k, \gamma^-}} \right\}.
\]
The function $f(q^-, p^+)$ generates a Lagrangian submanifold $L_f \subset S^+ \times (-S^-)$, and (5) means that the Hamiltonian $h^+(q^+, p^+) + h^-(q^-, p^-)$ vanishes on $L_f$. 
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The function $f(q^-, p^+)$ generates a Lagrangian submanifold $L_f \subset S^+ \times (-S^-)$, and (5) means that the Hamiltonian $h^+(q^+, p^+) + h^-(q^-, p^-)$ vanishes on $L_f$.

If $Y^- = \emptyset$ then $L_f$ is a Lagrangian submanifold in $S = S^+$. Composition of morphisms is the composition of Lagrangian correspondences.
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The codimension of a boundary stratum in the cylindrical case is equal to the number of components different from trivial cylinders. For manifolds with cylindrical ends it is equal to the number of non-trivial components in the cylindrical stories of the building. Thus in all cases the codimension 1 strata of the boundary consist only of 2-story buildings.
A symplectic manifold \((X, \omega)\) is a manifold, locally modeled on \((\mathbb{R}^{2n}, dp \wedge dq)\). Equivalently, according to the Darboux theorem, \((X, \omega)\) is just a manifold with a non-degenerate closed differential 2-form.

An almost complex structure \(J : TX \to TX, J^2 = -\text{Id}\), is called compatible with \(\omega\) if on each tangent space \(\omega\) and \(J\) relate to each other as the standard symplectic form \(dp \wedge dq\), and the standard complex structure do on \(\mathbb{R}^{2n} = \mathbb{C}^n\). In other words, if

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\omega(JV_1, V_2) - i\omega(V_1, V_2)
\]

is a positive definite Hermitian form on \(TX\). Given \(\omega\), one can always find a compatible \(J\) (but not the other way around). Moreover, this choice is unique up to homotopy.
There are 4 different ways to glue the two surfaces along their matching ends, i.e., the ends denoted by $p$'s and $q$'s with the same index. These 4 ways correspond to 4 terms in the composition formula for differential operators:

$$(\hbar^{-1} p_1 p_2 p_3) \circ (\hbar^{-1} q_1 q_2 p_1) = p_1 p_3 + \hbar^{-2} q_1 q_2 p_1^2 p_2 p_3 + \hbar^{-1} q_1 p_1^2 p_3 + \hbar^{-1} q_2 p_1 p_2 p_3.$$
An appearance of the coefficient $k$ in the commutator
$[\rho_{k,\gamma}, q_{k,\gamma}] = k\hbar$ (and the Poisson bracket
$\{\rho_{k,\gamma}, q_{k,\gamma}\} = k$ in the rational case)
corresponds to the fact that there are $k$ distinct ways of gluing $k$-multiple orbits.
Given any set $\Theta = (\theta_1, \ldots, \theta_k)$ of (not necessarily closed) differential forms on $Y$, and sets $\Gamma_-, \Gamma_+$ of periodic orbits of $R$ we define their correlator

$$\langle \theta_1, \ldots, \theta_k \rangle_{g, k}^{\Gamma_-, \Gamma_+} = \int_{\overline{M}_{g, k}^{\Gamma_-, \Gamma_+(Y, J)/R}} \ev_1^*(\theta_1) \wedge \cdots \wedge \ev_k^*(\theta_k),$$
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then organize them into an element of $\frac{1}{h} W$:

$$\langle \theta_1, \ldots, \theta_k \rangle_k = \sum_{\Gamma_-, \Gamma_+} \langle \theta_1, \ldots, \theta_k \rangle_{g,k}^{\Gamma_-, \Gamma_+} q^{\Gamma_+} p^{\Gamma_- h^g - 1}.$$
Given two sets of forms $\Theta = (\theta_1, \ldots, \theta_k)$ and $\Upsilon = (\upsilon_1, \ldots, \upsilon_m)$ we have $H^{\Theta \cup \Upsilon}(t, 0) = H^\Theta(t)$, where $t = (t_1, \ldots, t_k)$. In particular, $H^\Theta(0) = H^\Theta = H$. 
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"MASTER EQUATION":

$$H^{\Theta \cup d\Theta} \circ H^{\Theta \cup d\Theta} = DH^{\Theta \cup d\Theta}, \quad D = \sum_{1}^{k} t_i \frac{\partial}{\partial s_i},$$

where the variables $s = (s_1, \ldots, s_k)$ correspond to forms $d\Theta = (d\theta_1, \ldots, d\theta_k)$.
Given two sets of forms \( \Theta = (\theta_1, \ldots, \theta_k) \) and \( \Upsilon = (\upsilon_1, \ldots, \upsilon_m) \) we have \( H^{\Theta \cup \Upsilon}(t, 0) = H^\Theta(t) \), where \( t = (t_1, \ldots, t_k) \). In particular, \( H^\Theta(0) = H^\emptyset = H \).

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H^{\Theta \cup \Upsilon \cup d\Theta} \circ H^{\Theta \cup \Upsilon \cup d\Theta} = DH^{\Theta \cup \Upsilon \cup d\Theta}, \quad D = \sum_{i=1}^{k} t_i \frac{\partial}{\partial s_i},
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where the variables \( s = (s_1, \ldots, s_k) \) correspond to forms \( d\Theta = (d\theta_1, \ldots, d\theta_k) \). In particular, if \( d\Theta = 0 \) then we have

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Given two sets of forms $\Theta = (\theta_1, \ldots, \theta_k)$ and $\Upsilon = (\upsilon_1, \ldots, \upsilon_m)$ we have $H^{\Theta \cup \Upsilon}(t, 0) = H^\Theta(t)$, where $t = (t_1, \ldots, t_k)$. In particular, $H^{\Theta}(0) = H^\Theta = H$.

"Master Equation":

$$H^{\Theta \cup \Delta \Theta} \circ H^{\Theta \cup \Delta \Theta} = DH^{\Theta \cup \Delta \Theta}, \quad D = \sum_{1}^{k} t_i \frac{\partial}{\partial s_i}, \quad (6)$$

where the variables $s = (s_1, \ldots, s_k)$ correspond to forms $d\Theta = (d\theta_1, \ldots, d\theta_k)$. In particular, if $d\Theta = 0$ then we have

$$H^\Theta(t) \circ H^\Theta(t) = 0. \quad (7)$$

In many interesting examples (e.g. in presence of an $S^1$-symmetry) $H(0) = 0$. Hence, by differentiating (7) in $t$-variables we get $[H_i, H_j] = 0$, where $H_i = \frac{\partial H}{\partial t_i}(0)$, i.e. all $H_j$ are commuting differential operators.
Symplectic Field Theory can be further upgraded to include descendent variables, in a similar vein, as it is done in the Gromov-Witten theory. In particular, $H^\Theta$ becomes a function of infinitely many variables $T = (t_{ij}), i = 1, \ldots, k, j \geq 0$. Differentiating the identity $H^\Theta(T) \circ H^\Theta(T) = 0$, we observe existence of infinitely many commuting differential operators.

For instance, for $Y = S^1$, $\Theta = \{d\varphi\}$, one gets an infinite sequence of commuting integrals $H_j, j = 0, \ldots$, for the quantized Burgers hierarchy

$$\frac{du}{dt_{k+2}} = \frac{1}{k!} u^k \frac{du}{dx}, \quad k = 0, 1, \ldots$$

on the space of functions on $S^1$. 
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on the space of functions on $S^1$.

For $Y = S^3$ and $\Theta = \{\text{harmonic volume form}\}$, one gets commuting integrals of the quantized dispersionless Toda hierarchy which includes the equation $\ddot{u} = -(\exp(u-ix))_{xx}$, already appeared above in the problem of enumerating rational curves in $CP^2$. 