# Knots and Dynamics

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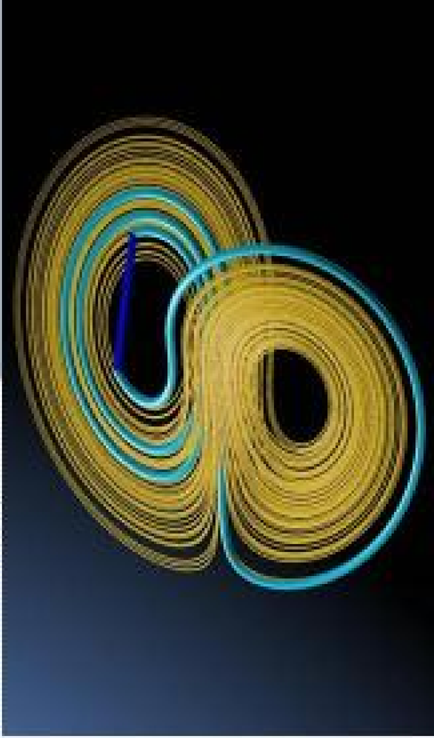
# Schwarzman-Sullivan-Thurston etc.

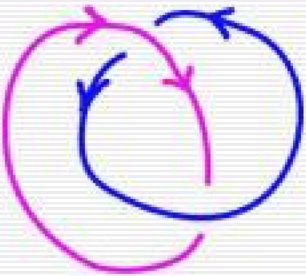
*One should think of a measure preserving vector field as « an asymptotic cycle »*



$$k(T,x) = \left\{ x \xrightarrow{\phi \text{ orbit}} \phi^T(x) \xrightarrow{\text{segment}} x \right\}$$

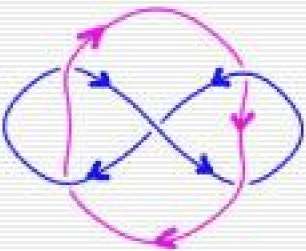
$$Flow \approx \text{"}\lim_{T\to\infty}\text{"} \int \frac{1}{T} k(T,x)\, d\mu(x)$$

# *Example : helicity as asymptotic linking number*



Linking # = +1

+1



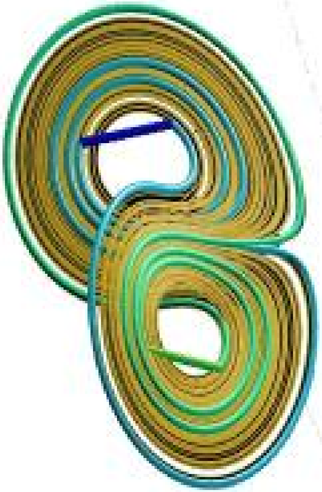
Linking # = 0

-1

# *Example : helicity as asymptotic linking number*

**Theorem** (Arnold) *Consider a flow in a bounded domain in* $\mathbf{R}^3$ *preserving an ergodic probability measure* $\mu$ *(not concentrated on a periodic orbit). Then, for* $\mu$*-almost every pair of points* $x_1, x_2$*, the following limit exists and is independent of* $x_1, x_2$

$$Helicity = \lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} Link(k(T_1, x_1), k(T_2, x_2))$$



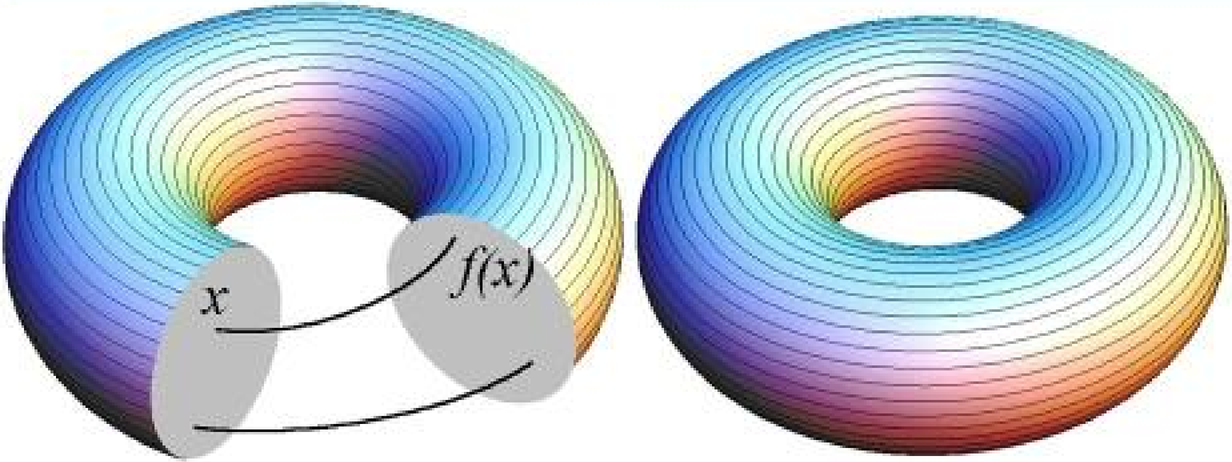
# Open question (Arnold):
# Is *Helicity* a topogical invariant ?

*Let $\phi_1^t$ and $\phi_2^t$ be two smooth flows preserving $\mu_1$ and $\mu_1$.*
*Assume there is an orientation preserving homeomorphism*
*$h$ such that $h \circ \phi_1^t = \phi_2^t \circ h$ and $h_* \mu_1 = \mu_2$.*
*Does it follow that*

*$Helicity(\phi_1^t) = Helicity(\phi_2^t)$ ?*

http://www.math.rug.nl/~veldman/movies/dns-large.mpg

**Suspension**: an area preserving diffeomorphim of the disk defines a volume preserving vector field in a solid torus.




Invariants on Diff ($\mathbf{D}^2$,*area*)?

**Theorem** (Calabi): *There is a non trivial homomorphism*

$Cal : \mathrm{Diff}^{\infty}(\mathrm{D}^2, \partial \mathrm{D}^2, area) \to \mathbf{R}$

**Theorem** (Banyaga): *The Kernel of $Cal$ is a simple group.*

**Theorem** (Gambaudo-G): $Cal(f)$ = *Helicity*(Suspension $f$)

**Theorem** (Gambaudo-G): *$Cal$ is a topological invariant.*

**Corollary** : *Helicity is a topological invariant for those flows which are suspensions.*

# A definition of Calabi's invariant (Fathi)

$f \in \mathrm{Diff}^{\infty}(\mathrm{D}^2, \partial\mathrm{D}^2, area)$

$f_t \quad (t \in [0,1])$

$f_0 = Id \qquad f_1 = f$

$$Cal(f) = \iint Var_{t=0}^{t=1}\, Arg\big(f_t(x) - f_t(y)\big)\, dx\, dy$$

Open question (Mather):

*Is the group* Homeo($\mathbf{D}^2$, $\partial\mathbf{D}^2$, *area*) *a simple group?*

- Can one extend $Cal$ to homeomorphisms?

- Good candidate for a normal subgroup:
the group of « hameomorphisms » (Oh). Is it non trivial?

# More invariants on the group Diff($\mathbf{D}^2$, $\partial\mathbf{D}^2$, area)?

No homomorphism besides Calabi's…

**Quasimorphisms** $\chi : \Gamma \to \mathbf{R}$ $\quad |\chi(\gamma_1\gamma_2) - \chi(\gamma_1) - \chi(\gamma_2)| \leq Const$

Homogeneous if $\chi(\gamma^n) = n\chi(\gamma)$

$\Gamma$ non abelian free group

↓

Many non trivial homogeneous quasimorphisms (Gromov, …)

*Abelian groups or $SL(n,\mathbf{Z})$ for $n>2$)*

↓

**No** non trivial (Trauber, Burger-Monod)

**Theorem** (Gambaudo-G) *The vector space of homogeneous quasimorphisms on* Diff($\mathbf{D}^2$, $\partial\mathbf{D}^2$, *area*) *is infinite dimensional.*

One idea: use **braids** and quasimorphisms on braid groups.

$$(x_1, x_2, \dots, x_n) \mapsto b(x_1, x_2, \dots, x_n) \in \mathrm{B}_n \xrightarrow{\chi} \mathbf{R}$$

Average over *n*-tuples of points in the disk

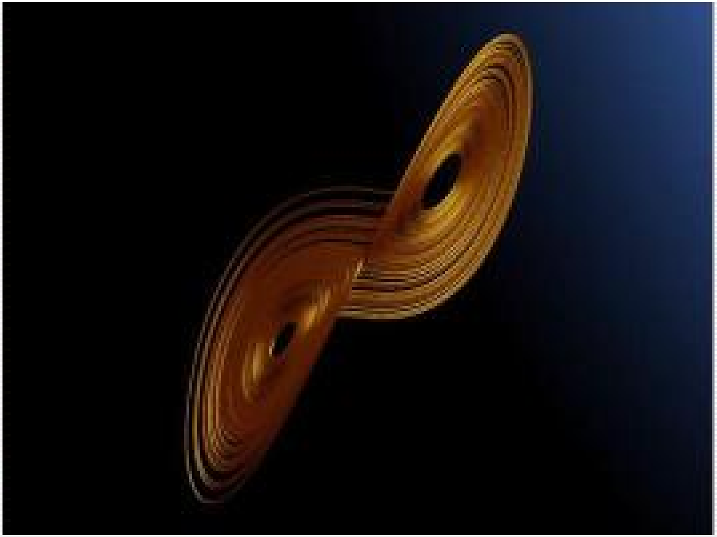
Example: *n*=2. Calabi homomorphism.

# Lorenz equation (1963)

$$\frac{dx}{dt} = 10(y - x)$$

$$\frac{dy}{dt} = 28x - y - xz$$

$$\frac{dz}{dt} = xy - \frac{8}{3}z$$



# More quasimorphims…

**Theorem** (Entov-Polterovich) :
*There exists a « Calabi quasimorphism »* $\chi : \mathrm{Diff}_0(\mathbf{S}^2, area) \to \mathbf{R}$
*such that* $\chi(f) = Cal(f_{|D})$
*when the support of f is in a disc D with area < 1/2 area(**S**[2]).*

**Theorem** (Py) :
*If* $\Sigma$ *is a compact orientable surface of genus g > 0,*
*there is a « Calabi quasimorphism »* $\chi : \mathrm{Ham}(\Sigma, area) \to \mathbf{R}$
*such that* $\chi(f) = Cal(f_{|D})$
*when the support of f is contained in some disc* $D \subset \Sigma$.

# Questions:

- *Can one define similar invariants for volume preserving flows in 3-space, in the spirit of « $Cal(f)$ = Helicity(suspension f ) »?*
- *Does that produce topological invariants for smooth volume preserving flows?*
- *Higher dimensional topological invariants in symplectic dynamics?*
- etc.

# Example : *the modular flow on* SL(2,**R**)/SL(2,**Z**)

- Space of lattices in $\mathbf{R}^2$ of area 1.

$$\Lambda \approx \mathbf{Z}^2 \subset \mathbf{R}^2$$

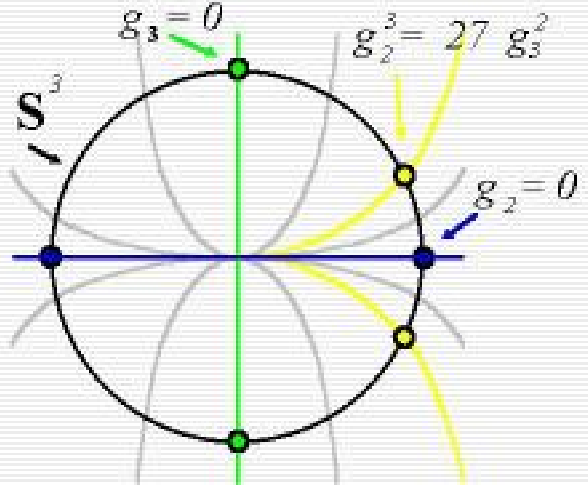
$$area(\mathbf{R}^2 / \Lambda) = 1$$

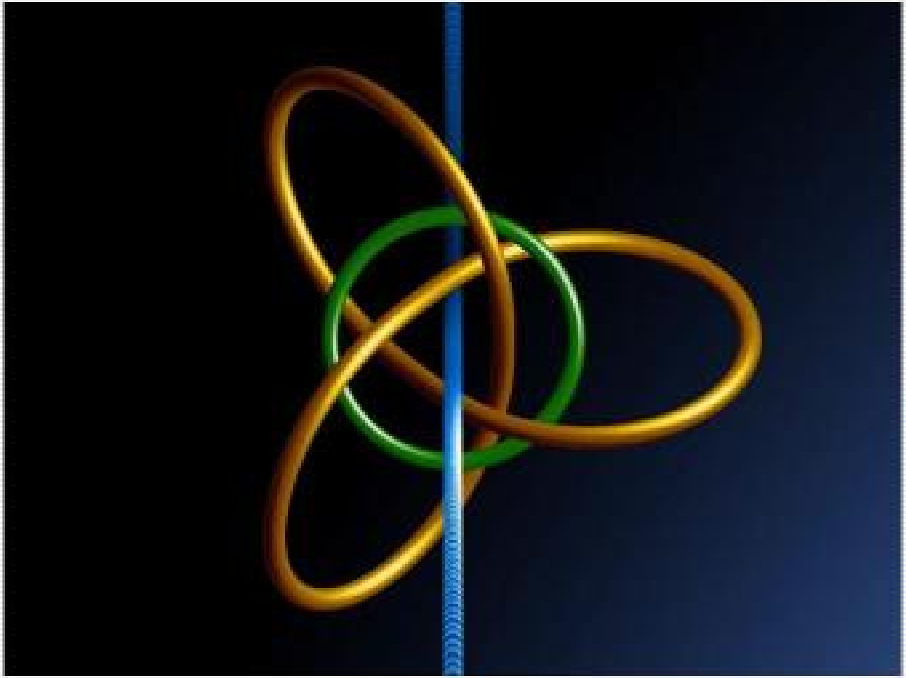
# **Topology:** SL(2,**R**)/SL(2,**Z**) is homeomorphic to the complement of the trefoil knot in the 3-sphere

$$g_2(\Lambda) = 60\sum_{\omega\in\Lambda-\{0\}}\omega^{-4} \in \mathbf{C} \qquad g_3(\Lambda) = 140\sum_{\omega\in\Lambda-\{0\}}\omega^{-6} \in \mathbf{C}$$

$$\Lambda \subset \mathbf{R}^2 \approx \mathbf{C} \mapsto \left(g_2(\Lambda), g_3(\Lambda)\right) \in \mathbf{C}^2 - \left\{g_2{}^3 = 27 g_3{}^2\right\}$$

$$area(\mathbf{R}^2 / \Lambda) = 1$$





# Dynamics

On lattices $\phi^t(\Lambda) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}(\Lambda)$

# Periodic orbits

$$A \in SL(2,\mathbf{Z}) \qquad A(\mathbf{Z}^2)=\mathbf{Z}^2$$

$$PAP^{-1} = \pm\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \qquad \Lambda = P(\mathbf{Z}^2)$$

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}(\Lambda) = \Lambda$$

QuickTime™ and a
H.264 decompressor
are needed to see this picture.

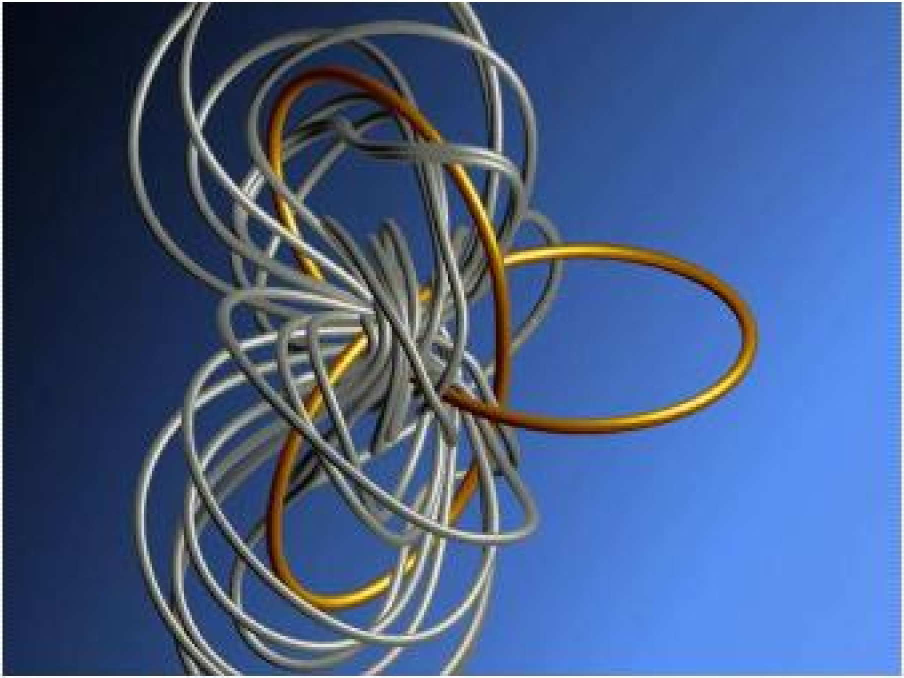
- Conjugacy classes of hyperbolic elements in PSL(2,**Z**)
- Closed geodesics on the modular surface D/PSL(2,**Z**)
- Ideal classes in quadratic fields
- Indefinite integral quadratic forms in two variables.
- Continuous fractions etc.

# Each hyperbolic matrix A in PSL(2,Z) defines a periodic orbit in SL(2,**R**)/SL(2,**Z**), hence a closed curve $k_A$ in the complement of the trefoil knot.

Questions :

1) What kind of knots are the « modular knots » $k_A$ ?

2) Compute the linking number between $k_A$ and the trefoil.



## **Theorem:** *The linking number between* $\boldsymbol{k_A}$ *and the trefoil is equal to R(A) where R is the « Rademacher function ».*

Dedekind eta-function $\eta(\tau) = \exp(i\pi\tau/12)\prod_{n\geq 1}(1-\exp(2i\pi n\tau))$ ; $\Im(\tau)>0$

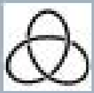
$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)^{24} = \eta(\tau)^{24}(c\tau+d)^{12} \quad ; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z})$$

$$24(\log\eta)\left(\frac{a\tau+b}{c\tau+d}\right) = 24(\log\eta)(\tau) + 6\log(-(c\tau+d)^2) + 2i\pi \mathrm{R}(A)$$

$\mathrm{R} : SL(2,\mathbf{Z}) \to \mathbf{Z}$ This is a *quasimorphism*.

# **Birman-Williams:** Periodic orbits are ***knots***.

Topological description of the Lorenz attractor

# « Proof » that R(*A*) = *Linking*($k_A$,  )

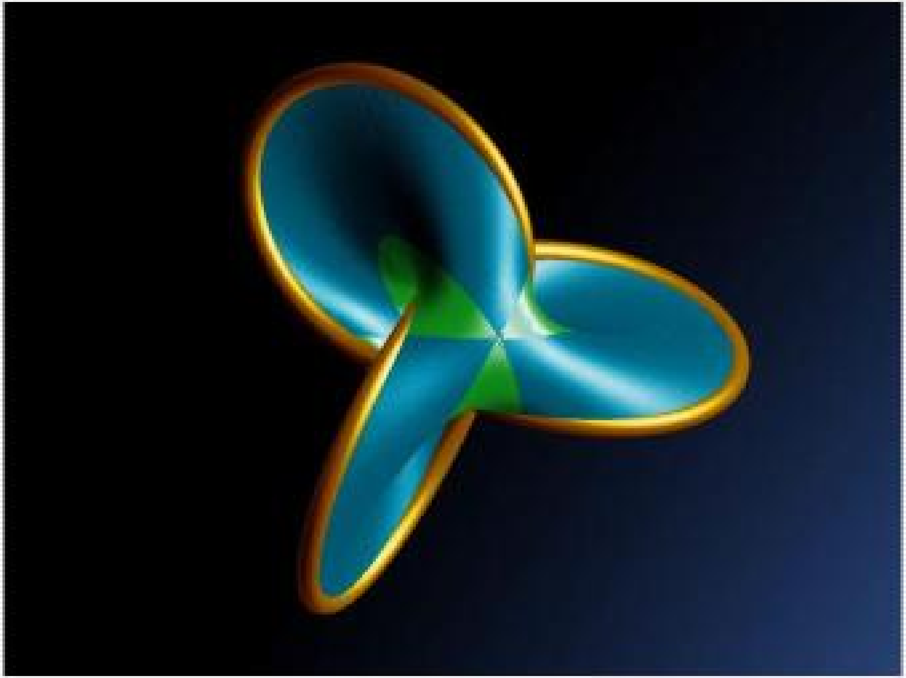
Jacobi proved that

$$(g_2^3 - 27g_3^2)(\mathbf{Z} + \tau\mathbf{Z}) = (2\pi)^{12}\,\eta(\tau)^{24}$$

$g_2^3 - 27g_3^2 = 0$ is the trefoil knot

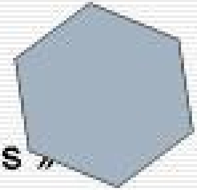
$$\log(z) = \log|z| + i\ \mathrm{Arg}(z)$$

$(g_2^3 - 27g_3^2)(\Lambda) \in \mathbf{R}^+$ is a Seifert surface

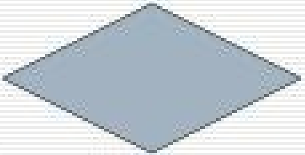


# Theorem:
## *Modular knots (and links) are the same as Lorenz knots (and links)*

**Step 1:** find some template inside SL(2,**R**)/SL(2,**Z**) which looks like the Lorenz template.

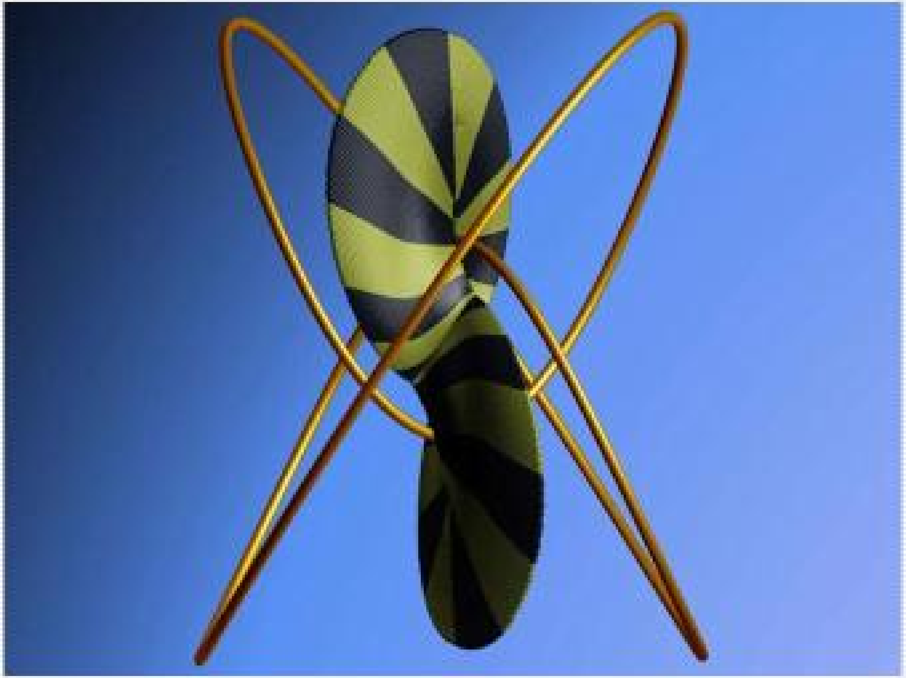


Look at « regular hexagonal lattices »

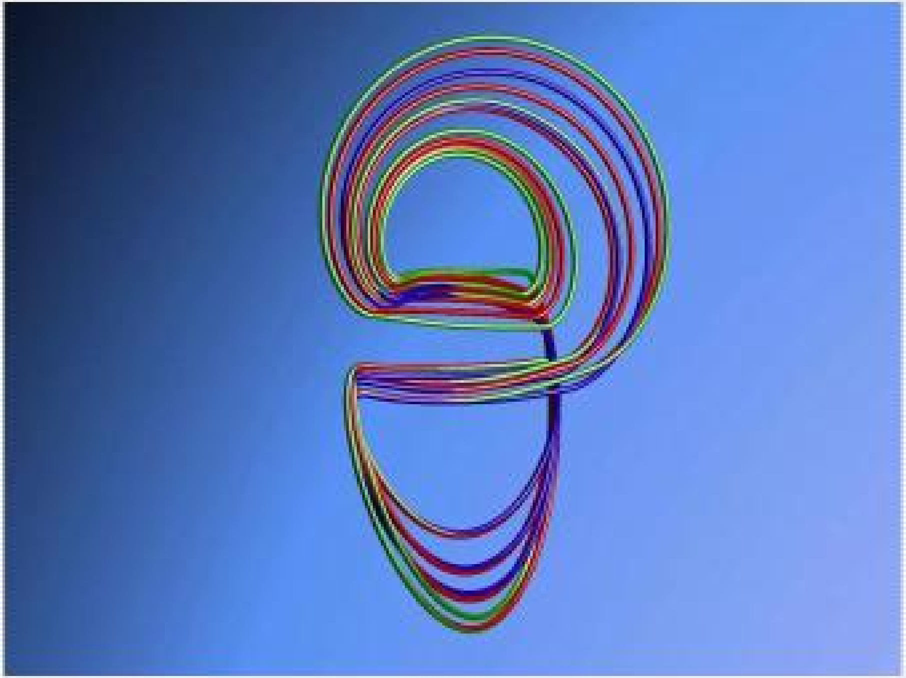


and lattices with horizontal rhombuses as fundamental domain with angle between 60 and 120 degrees.

Make it thicker by pushing along the unstable direction.



**Step 2** : deform lattices to make them approach the Lorenz template.



# Further developments?

•From modular dynamics to Lorenz dynamics and vice versa?

• For instance, « modular explanation » of the fact that all Lorenz links are fibered?

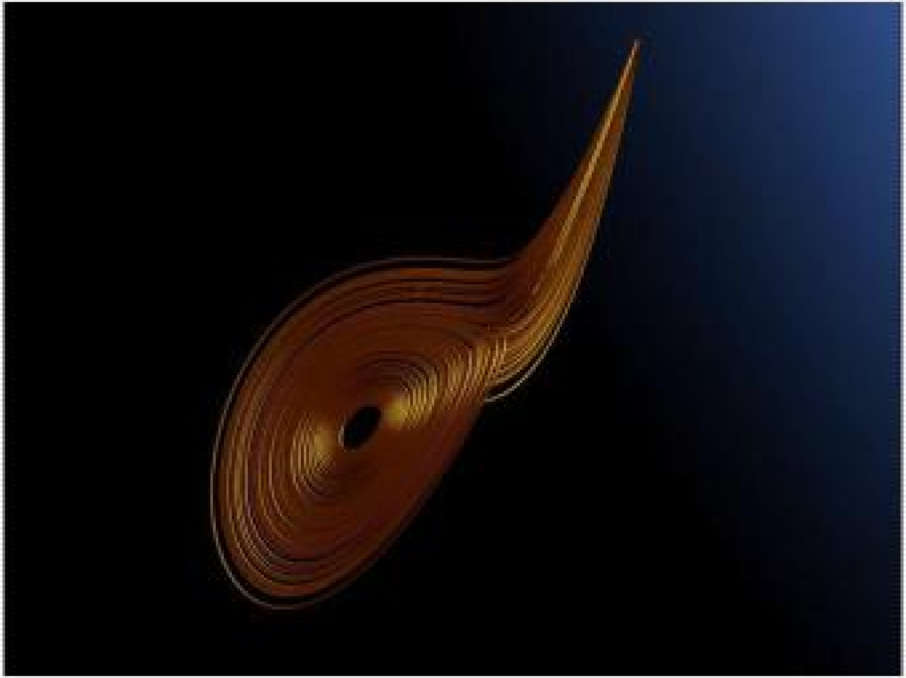
Many thanks to Jos Leys !

Mathematical Imagery : http://www.josleys.com/

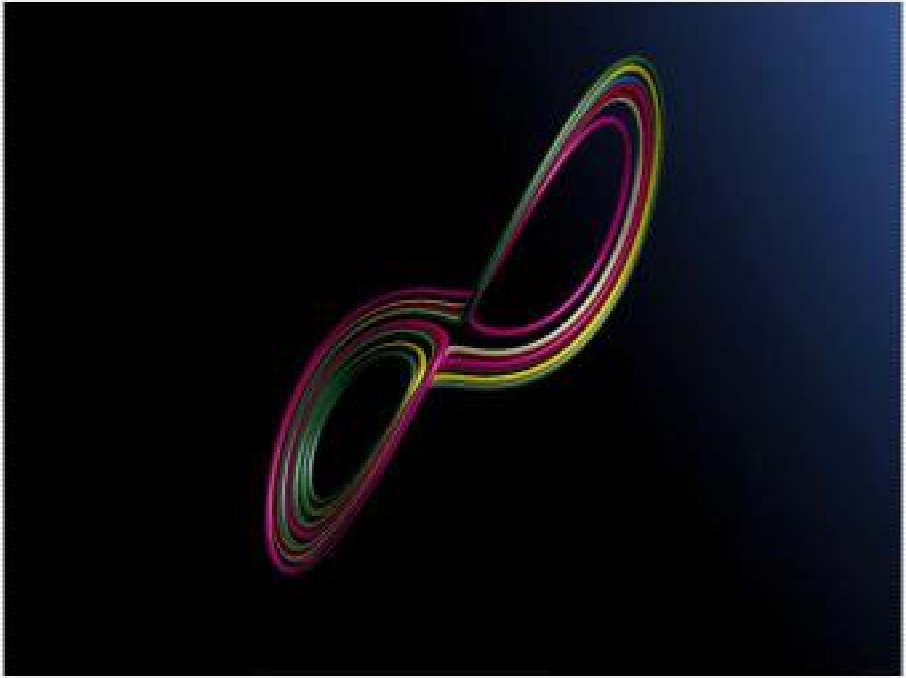
*« A mathematical theory is not to be considered complete until you made it so clear that you can explain it to the man you meet on the street »*

*« For what is clear and easily comprehended attracts and the complicated repels us »*

**D. Hilbert**

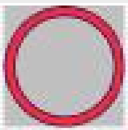


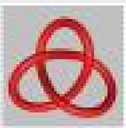
QuickTime™ et un
décompresseur H.264
sont requis pour visionner cette image.



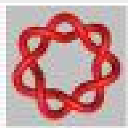
# Birman-Williams:
# Lorenz knots and links are very peculiar

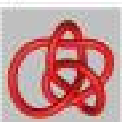
- Lorenz knots are prime
- Lorenz links are fibered
- non trivial Lorenz links have positive signature

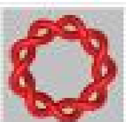

$0_1$ Unknot


$3_1$ trefoil

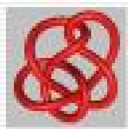

$5_1$Cinquefoil

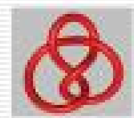

$7_1$


$8_{19}$ =T(4,3)


$9_1$


$10_{124}$=T(5,3)


$10_{132}$


$4_1$ Figure eight