Iwasawa theory
and
Generalizations

Zeta values $\leftrightarrow$ Arithmetic

?? ?? ?? ?? ??
(3) Iwasawa theory (195*)

p-adic Riemann zeta function
(zeta side)

\[ \text{Cl} \left( \mathbb{Q}(5_m) \right) \{p\} \]
with action of \( \text{Gal} \left( \mathbb{Q}(5_m)/\mathbb{Q} \right) \)
(arithmetic side)
Theorem (Herbrand–Ribet)

Let $r \in \mathbb{Z} < 0$, odd. Then:

\[ p \mid S(r) \]

\[ \iff \mathbb{Z}/p\mathbb{Z}(r) \subset \text{Cl}(\mathbb{Q}(5_p)) \text{ as } \text{Gal}(\mathbb{Q}(5_p)/\mathbb{Q})\text{-module} \]

Here $\mathbb{Z}/p\mathbb{Z}(r)$ is $\mathbb{Z}/p\mathbb{Z}$ on which

\[ \sigma_a \in \text{Gal}(\mathbb{Q}(5_p)/\mathbb{Q}) \text{ acts as } a^r \in (\mathbb{Z}/p\mathbb{Z})^\times \]

\[ \sigma_a(5_p) = 5_p^a \]
\[ \mathbb{Z}/p\mathbb{Z}(r) \subset \mathfrak{a}(\mathbb{Q}(\sqrt{p})) \text{ as } \text{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q}) \text{-module} \]

Here \( \mathbb{Z}/p\mathbb{Z}(r) \) is \( \mathbb{Z}/p\mathbb{Z} \) on which

\( \sigma_a \in \text{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q}) \) acts as \( a^r \in (\mathbb{Z}/p\mathbb{Z})^\times \)

\( \left( \frac{\mathfrak{a}}{\sigma_a} \right) \mathfrak{a} = (\mathbb{Z}/p\mathbb{Z})^\times \)

\[
\begin{align*}
\begin{bmatrix}
\mathfrak{a} \\
\sigma_a
\end{bmatrix}
\begin{bmatrix}
\mathfrak{a} \\
\sigma_a
\end{bmatrix}
= (\mathbb{Z}/p\mathbb{Z})^\times
\end{align*}
\]

\( \sigma_a(\sqrt{p}) = \sqrt{p}^a \)

**Example**

For \( p = 691 \),

\[ \mathbb{Z}/p\mathbb{Z}(\sqrt{-11}) \subset \text{Cl}(\mathbb{Q}(\sqrt{p})) \]

as \( \text{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q}) \text{-module} \)
Iwasawa main conjecture proved by Mazur-Wiles (1984)

\[ I_1 \cdot 5_p\text{-adic} = \text{Char}_\Lambda(X) \]
(zeta side) (arithmetic side)

Here:

\[ 5_p\text{-adic} = p\text{-adic Riemann zeta function} \]

\[ X = \text{Hom}(\lim_{\to} \text{Cl}(\mathbb{Q}(5_p))^{-}, \mathbb{Q}/\mathbb{Z}_p) \]

\[ I_1 = \text{Ker}(\Lambda \to \mathbb{Z}_p; \text{value at } 1) \]
II Elliptic curves

$E/Q$ elliptic curve

good ordinary reduction at $p$

$(E(\overline{\mathbb{F}_p})[p] = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{ordinary} \\ 0 & \text{super-singular} \end{cases})$

$\mathbb{Q}^{cyc} > \mathbb{Q} \quad G^{cyc} = \text{Gal}(\mathbb{Q}^{cyc}/\mathbb{Q}) \cong \mathbb{Z}_p$

$K$ : imaginary quadratic field in which $p$ splits

$K_{00} > K \quad G = \text{Gal}(K_{00}/K) \cong \mathbb{Z}_p$

$\Lambda^{cyc} = \mathbb{Z}_p[[G^{cyc}]] = \varprojlim_n \mathbb{Z}_p[\text{Gal}(\mathbb{Q}_n/\mathbb{Q})] \cong \mathbb{Z}_p[[T]]$

$\Lambda = \mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\text{Gal}(K_n/K)] \cong \mathbb{Z}_p[[T_1, T_2]]$
Zeta side \quad (p\text{-adic } L\text{-functions})

\[ L_p(E, \mathbb{Q}^{cyc}/\mathbb{Q}) \in \Lambda^{cyc} \]

\[ L_p(E, K_\infty/K) \in \Lambda \]

which \( p \)-adically interpolates the complex zeta values.

\[ \frac{L(E, 1, x)}{\text{period}} \in \mathbb{Q} \quad (x: G^{cyc} \to \overline{\mathbb{Q}}^x) \]

\[ \frac{L(E_k, 1, x)}{\text{period}} \in \overline{\mathbb{Q}} \quad (x: G \to \overline{\mathbb{Q}}^x) \]

\[ (L(E, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \Rightarrow L(E, s, x) = \sum_{n=1}^{\infty} \frac{a_n X(n^s)}{n^x}) \]
Arithmetic side (Dual Selmer groups)

\[ X^{\text{cyc}} = \text{Hom} \left( \varinjlim_n \text{Sel}(E/Q_n), \mathbb{Q}_p/\mathbb{Z}_p \right) ; \Lambda^{\text{cyc}} \text{-module} \]

\[ X = \text{Hom} \left( \varinjlim_n \text{Sel}(E/K_n), \mathbb{Q}_p/\mathbb{Z}_p \right) ; \Lambda \text{-module} \]

\[ 0 \to E(F) \otimes \mathbb{Q}/\mathbb{Z} \to \text{Sel}(E/F) \to \text{III}(E/F) \to 0 \]

Selmer group \( \text{conjectured to be finite} \)

For \( R = \Lambda^{\text{cyc}} \) or \( \Lambda \),

\[ M : \text{f.g. torsion} \Lambda \text{-module} \]

\[ \Rightarrow M \sim R/(a_1) \oplus \cdots \oplus R/(a_n) \quad (a_i \neq 0) \]

\[ \text{Char}_R(M) := (\prod_{i=1}^n a_i) \subseteq R \]
Arithmetic side (Dual Selmer groups)

\[ X^{cyc} = \text{Hom}(\lim_{n} \text{Sel}(E/\mathbb{Q}_n), \mathbb{Q}_p/\mathbb{Z}_p) ; \wedge^{cyc} \text{-module} \]

\[ X = \text{Hom}(\lim_{n} \text{Sel}(E/K_n), \mathbb{Q}_p/\mathbb{Z}_p) ; \wedge \text{-module} \]

\[ (0 \to E(F) \otimes \mathbb{Q}/\mathbb{Z} \to \text{Sel}(E/F) \to \text{III}(E/F) \to 0 ) \]

\[ \uparrow \quad \text{Selmer group conjectured to be finite} \]

For \( R = \wedge^{cyc} \) or \( \wedge \),

\( M : \text{t.g. torsion } \wedge \text{-module} \)

\[ \Rightarrow M \sim R/(a_1) \oplus \cdots \oplus R/(a_n) \quad (a_i \neq 0) \]

\[ \text{Char}_R(M) := (\prod_{i=1}^{n} a_i) \subset R \]
\( 0 \rightarrow E(F) \otimes Q/Z \rightarrow Sel(E/F) \rightarrow \Sha(E/F) \rightarrow 0 \)

Selmer group conjectured to be finite

---

For \( R = \Lambda_{\text{cr}} \) or \( \Lambda \),
M: f. g. torsion \( \Lambda \)-module
\[ \Rightarrow M \cong R/(a_1) \oplus \cdots \oplus R/(a_n) \quad (a_i \neq 0) \]

\[ \text{Char}_R(M) := \left( \frac{n}{a_1} \cdot a_2 \right) \subseteq R \]

\( \text{Char}_R(M) \) is an \( R \)-module analogue of
\[ \#(M) \quad M: \text{finite abelian group.} \]
\[ M \cong Z/(a_1) \oplus \cdots \oplus Z/(a_n), \quad \#(M) = \left( \prod_{i=1}^{n} a_i \right) \]
History

(1) Euler

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]

\[ \zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} \quad (1735) \]

\[ \zeta(4) = \frac{\pi^4}{90} , \quad \zeta(6) = \frac{\pi^6}{945} , \cdots \]

\[ \zeta(r) \in \mathbb{Q}\pi^r \text{ if } r \in \mathbb{Z}_{>0} \text{ even} \]
• $X^{\text{cyc}}$ is a torsion $\Lambda^{\text{cyc}}$-module. (K)

From this, we can deduce

• $X$ is a torsion $\Lambda$-module.

Recent work of Skinner-Urban gives

**Theorem** Under a mild assumption, we have

$$\left( L_p(E, Q^{\text{cyc}}/\mathbb{Q}) \right) = \text{Char}_{\Lambda^{\text{cyc}}}(X^{\text{cyc}})$$

$$\left( L_p(E, K_{\infty}/K) \right) = \text{Char}_{\Lambda}(X)$$

This is a consequence of:

(1) $\text{Char}_{\Lambda^{\text{cyc}}}(X^{\text{cyc}}) \mid L_p(E, Q^{\text{cyc}}/\mathbb{Q})$
Recent work of Skinner-Urban gives

Theorem. Under a mild assumption, we have

\[
(L_p(E, Q^{\text{cyc}}/Q)) = \text{Char}_{\Lambda_{\text{cyc}}}^{\Lambda}(X^{\text{cyc}})
\]

\[
(L_p(E, K_{\infty}/K)) = \text{Char}_{\Lambda}(X)
\]

This is a consequence of:

1. \( \text{Char}_{\Lambda_{\text{cyc}}}^{\Lambda}(X^{\text{cyc}}) \parallel L_p(E, Q^{\text{cyc}}/Q) \) (under a mild assumption)

2. \( L_p(E, K_{\infty}/K) \parallel \text{Char}_{\Lambda}(X) \) (Skinner-Urban, under a mild assumption)
In the case $E$ has complex multiplication by the imaginary quadratic field $K$, Theorem was proved by Rubin.

For the Iwasawa theory of $(E, K^{\text{anti-cyc}}/K)$ (anti-cyclotomic Iwasawa theory of $E$)

\begin{align*}
Z_p & \quad K_{\infty} \quad Z_p \\
K^{\text{anti-cyc}} & \quad Z_p \\
Z_p & \quad K \quad Z_p
\end{align*}

Bertolini and Darmon proved

(arithmetic side) $|$ (zeta side)
Two methods in Iwasawa theory

(1) Euler system method to prove 
\[
\begin{align*}
& \text{(arithmetic side)} \text{ is torsion} \\
& \text{(arithmetic side)} \mid \text{(zeta side)}
\end{align*}
\]

(2) Modular form method to prove 
\[
\text{(zeta side)} \mid \text{(arithmetic side)}
\]

Case of classical Iwasawa main conj
1) Euler system method
to prove
\[
\{ \text{(arithmetic side)} \} \text{ is torsion}
\{ \text{(arithmetic side)} \} \mid \{ \text{zeta side} \}
\]

2) Modular form method
to prove
\[
\{ \text{zeta side} \} \mid \{ \text{arithmetic side} \}
\]

Case of classical Iwasawa main conj
The first proof by Mazur-Wiles was (2).
The second proof by Rubin was (1).
Recent work of \[ \text{urban} \] gives

**Theorem**  Under a \[ \text{assumption, we have} \]

\[
(L_p (E, Q^{\text{sys}}/Q)) = \text{Char}^\text{\text{sys}}_{\text{sys}} (X^{\text{\text{sys}}})
\]

\[
(L_p (E, K_{\infty}/K)) = \text{Char}_{\Lambda} (X)
\]

This is a consequence of:

1. \[ \text{Char}^\text{\text{sys}}_{\text{sys}} (X^{\text{\text{sys}}}) \parallel L_p (E, Q^{\text{\text{sys}}}/Q) \]
   \[ (K, \text{ under a mild assumption }) \]

2. \[ L_p (E, K_{\infty}/K) \parallel \text{Char}_{\Lambda} (X) \]
   \[ (\text{Skinner-Urban, under a mild assumption}) \]
(1) Euler system method

discovered by Kolyvagin

zeta function (analytic) too far arithmetic

incarnation → very near

Euler system (arithmetic (family))
Case of classical Iwasawa theory

Arithmetic incarnations of $\zeta(s)$, $L(s, \chi)$ are cyclotomic units

$$1 - \alpha \quad (\alpha: \text{root of } 1, \alpha \neq 1),$$

which are closely related to zeta values.

\[
\begin{align*}
\text{In } \mathbb{R}, \mathbb{C}: \\
-\log(1 - \alpha) &= \sum_{n=1}^{\infty} \frac{\alpha^n}{n} = \text{zeta value} \left. \sum_{n=1}^{\infty} \frac{\alpha^n}{n^s} \right|_{s=1} \\
\sum_{\chi(a) \neq 0} \chi(a) \log(1 - 5^a) &= -2L'(0, \chi)
\end{align*}
\]

\[
\sum_{a \in (\mathbb{Z}/N)^*} \chi(a) \log(1 - 5^a) = -2L'(0, \chi)
\]

for $\chi:(\mathbb{Z}/N)^* \to \mathbb{C}^*$, $\chi(-1) = 1$

\[
\text{In } \mathbb{Q}_p:
\]
$1 - \alpha$ \quad (\alpha : \text{root of 1, } \alpha \neq 1),
which are closely related to zeta values.

In $\mathbb{R}$, $\mathbb{C}$:

$$\log(1-\alpha) = \sum_{n=1}^{\infty} \frac{\alpha^n}{n} = \text{zeta value} \quad \sum_{n=1}^{\infty} \frac{\alpha^n}{n^s} \bigg|_{s=1}$$

$$\sum_{a \in (\mathbb{Z}/N)^x} \chi(a) \log |1-\xi_N^a| = -2 \zeta'(0, \chi)$$

for $\chi: (\mathbb{Z}/N)^x \to \mathbb{C}^x$, $\chi(-1) = 1$

In $\mathbb{Q}_p$:

$$(1-\xi_{p^n})_{n=1}$$

homomorphism of Kummer-Iwasawa

$p$-adic Riemann zeta function

- Coates-Wiles
- Coleman

$S(r)$ \quad r \in \mathbb{Z}_{\geq 0}$
Zeta functions enter the arithmetic world transforming themselves into Euler systems and produce the formula

(arithmetic side) | (zeta side)

Somewhere (?)
(ZetaLand ?)

- Zeta functions in $\mathbb{R}, \mathbb{C}$
- $p$-adic zeta functions in $\mathbb{Q}_p$
The study of zero was started by Euler.

Euler was happy that the difficult question of the sums of the series...
<table>
<thead>
<tr>
<th>Incarnation of</th>
<th>Cyclotomic units</th>
<th>Beilinson elements</th>
<th>Hasse points</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi(s)$</td>
<td>$\zeta(s)$</td>
<td>$L(E, s)$</td>
<td>$L(E, s)$</td>
</tr>
<tr>
<td>they live in</td>
<td>$\mathbb{Q}(\zeta_N)^X$</td>
<td>$K_2(\text{modular curve})$</td>
<td>Jacobian of modular curve $\mathcal{C}$ $K_0(\text{modular curve})$</td>
</tr>
<tr>
<td>$K_1(\mathbb{Q}(\zeta_N))$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>related in $\mathbb{R}, \mathbb{C}$ to</td>
<td>$L'(0, X)$</td>
<td>$L'(E, 0, X)$</td>
<td>$L'(E_K, 1, X)$</td>
</tr>
<tr>
<td>$\mathbb{Q}_p$</td>
<td>$L_p(E, \mathbb{Q}^{\text{cyc}}/\mathbb{Q})$</td>
<td>$L_p(E, \mathbb{K}^{\text{anticyc}}/\mathbb{K})$</td>
<td>$L_p(E, \mathbb{K}^{\text{anticyc}}/\mathbb{K})$</td>
</tr>
<tr>
<td>$\mathbb{Q}_p$</td>
<td>$\zeta_p$ function</td>
<td>$\mathcal{L}_p$</td>
<td>$\mathcal{L}_p$</td>
</tr>
</tbody>
</table>
How we can obtain

\[(\text{arithmetic side}) \div (\text{zeta side})]\n
in the case of classical Iwasawa theory.

The system of cyclotomic units produces
(via a procedure discovered by Kolyvagin)
principal ideals
as many as expected,
showing that
ideal class group \((= \frac{\{\text{fractional ideals}\}}{\{\text{principal ideals}\}})\)
is as small as expected;
that is,
(2) Modular form method

\[ S(s), L(s, \chi) \] is a zeta function of \( L(E, s) \).

It is a modular form of \( \begin{bmatrix} \text{GL}_1 \\ \text{GL}_2 \end{bmatrix} \).

Iwasawa theory of it can be studied by using the theory of modular forms of \( \begin{bmatrix} \text{GL}_2 \\ (\text{Mazur-Wiles}) \end{bmatrix} \).
$(ζ(s), L(s, X))$ is a zeta function of $L(E, s)$.

A modular form of $GL_1$ a modular form of $GL_2$.

Iwasawa theory of it can be studied by using the theory of modular forms of $GL_2$ (Mazur-Wiles).

$U(2, 2)$ (Skinner-Urban).
\[ S(r) = \begin{cases} \frac{-1}{2} & \text{if } r = 0 \\ 0 & \text{if } r \in \mathbb{Z}_{>0} \text{ even} \\ 2 - (m-1)! \frac{S(m)}{(2\pi i)^m} & \in \mathbb{Q} \text{ if } r \in \mathbb{Z}_{>0} \text{ odd (m = 1-r)} \end{cases} \]

\[ S(-1) = 1 + 2 + 3 + 4 + \ldots = 2 \cdot (2-1)! \frac{\pi^2/6}{(2\pi i)^2} = -\frac{1}{12} \]

\[ S(-3) = \frac{1}{2^3 \times 3 \times 5} \quad S(-5) = -\frac{1}{2^2 \times 3^2 \times 7} \]

\[ S(-7) = \frac{1}{2^4 \times 3 \times 5} \quad S(-9) = -\frac{1}{2^2 \times 3^2 \times 11} \]
Review: How Ribet’s theorem

\[ p \mid \zeta(n) \Rightarrow \mathbb{Z}/p\mathbb{Z}(n) \subset \text{Cl}(\mathbb{Q}(\sqrt[p]{5})) \]

as \( \text{Gal}(\mathbb{Q}(\sqrt[p]{5})/\mathbb{Q}) \)-module

is proved

This is a part of

(\zeta \text{ side}) \mid (\text{arithmetic side})

Three key points:

1) Riemann zeta values appear as constant terms of
Review: How Kitaoka showed

\[ \mathbb{Z}/p \mathbb{Z}(r) \subseteq \text{GL}(\mathbb{Q}(\sqrt{p})) \]

as Gal(\mathbb{Q}(\sqrt{p})/\mathbb{Q})-module

is proved

This is a part of

(zeta side) \mid (arithmetic side)

Three key points:

(i) Riemann zeta values appear as constant terms of Eisenstein series of GL2

\[ \zeta(s) = \frac{1}{\zeta(2s)} \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} \]
This is a part of 
\( \text{(zeta side)} \mid \text{(arithmetic side)} \)

---

Three key points:

(i) Riemann zeta values appear as constant terms of Eisenstein series of \( GL_2 \)

\[
E_{k-r} = \frac{\zeta(r)}{2} + \sum_{n=1}^{\infty} \sigma_{-r}(n) q^n \\
E_{k-r} = \frac{\zeta(r)}{2} + \sum_{n=1}^{\infty} \sigma_{-r}(n) q^n
\]

\( \sigma_m(n) = \sum_{d \mid n} d^m \)
This is a part of
(zeta side) \mid (arithmetic side)

Three key points:

(i) Riemann zeta values appear as constant terms of
    Eisenstein series of GL2

\[ E_{1-r} = \frac{\zeta(r)}{2} + \sum_{n=1}^{\infty} \sigma_{-r}(n) q^n \]

\[ (\sigma_m(n) := \sum_{d|n} d^m) \]
(ii) An eigen modular form at $GL_2$ produces a Galois representation
\[ \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2 \text{ (p-adic field)} \]

(Langlands correspondence
modular form $\leftrightarrow$ Galois rep)

(iii) Ideal class group
\[ = \{ \text{Extension classes of finite Galois representations} \} \]
\[ \mathbb{Z}/p\mathbb{Z} \langle \mathfrak{m} \rangle \subset \text{Cl}(Q(\sqrt{p})) \text{ as Gal}(Q(\sqrt{p})/Q) \text{-module} \]
\[ \Leftrightarrow \exists \text{ extension} \]
\[ \frac{\mathbb{Z}}{p\mathbb{Z}} \rightarrow \mathbb{Z}/p\mathbb{Z} ightarrow 0 \]
$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2$ (p-adic field)

(Langlands correspondence
modular form $\leftrightarrow$ Galois rep)

(iii) Ideal class group

$= \{ \text{Extension classes of finite Galois representations} \}$

$\mathbb{Z}/p\mathbb{Z}(\tau) \subset \text{Cl}(\mathbb{Q}(\sqrt[p]{\tau}))$ as $\text{Gal}(\mathbb{Q}(\sqrt[p]{\tau})/\mathbb{Q})$-module

$\leftrightarrow$ $\exists$ extension

$0 \rightarrow \mathbb{Z}/p\mathbb{Z}(\tau) \rightarrow \ast \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0$

of representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{Z}/p\mathbb{Z}$

(-) non-split

(-) unramified outside p
$P | 5(r)$

by (i) $\downarrow$

$E_{r-1} \mod P$ has no constant term

$\downarrow$

$E_{r-1} \equiv f \mod P \Leftrightarrow f$: eigen cusp form

Example

$691 | 5(-11)$

$E_{12} \equiv \Delta \mod 691$ (Ramanujan's congruence)

$\Delta = q \prod_{n=1}^{\infty} (1-q^n)^{24}$ eigen cusp form
\[ p \mid 5(r) \]

by (i) \[ \downarrow \]

\[ E_{1-r} \equiv f \mod p \exists f : \text{eigen cusp form} \]

Example

\[ 691 \mid 5(-11) \]

\[ E_{12} \equiv \Delta \mod 691 \]

\[ \Delta = \prod_{n=1}^{\infty} (1-q^n)^{24} \]

(Ramanujan's congruence)
by (ii) \[\downarrow\]

\[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{p^t} \text{GL}_2(\mathbb{Z}_p) \to \text{GL}_2(\mathbb{Z}/p\mathbb{Z}).\]

This representation over \(\mathbb{Z}/p\mathbb{Z}\) is an extension

\[0 \to \mathbb{Z}/p\mathbb{Z}(r) \to * \to \mathbb{Z}/p\mathbb{Z} \to 0\]

satisfying (\(*\)).

by (iii) \[\downarrow\]

\[\mathbb{Z}/p\mathbb{Z}(r) \subset \text{Cl}(\mathbb{Q}(5p))\]
as a \(\text{Gal}(\mathbb{Q}(5p)/\mathbb{Q})\)-module.
Key points in the method of Skinner-Urban

(i) $L(E_k, 1, \chi)$ appear as constant terms of Eisenstein series of $U(2, 2)$

(ii) An eigen modular form of $U(2, 2)$ produces a Galois representation

(iii) $\text{Sel}_{p^n}(E/F)$

= \{ extensions $E[p^n] \to * \to \mathbb{Z}/p^n\mathbb{Z} \to 0$ \}
2 \cdot (m-1)! \frac{\zeta(m)}{(2\pi i)^m} \in \mathbb{Q} \text{ if } m \in \mathbb{Z}_{>0} \text{ odd (} m = 1 - \gamma \text{)}

\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = 2 \cdot (2-1)! \frac{\pi^2/6}{(2\pi i)^2} = -\frac{1}{12}

\zeta(-3) = \frac{1}{2^3 \cdot 3 \cdot 5}
\zeta(-5) = -\frac{1}{2^2 \cdot 3^2 \cdot 7}

\zeta(-7) = \frac{1}{2^4 \cdot 3 \cdot 5}
\zeta(-9) = -\frac{1}{2^2 \cdot 3 \cdot 11}

\zeta(-11) = \frac{691}{2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13}
Key points in the method of Skinner-Urban

(i) \( L(E_k, 1, \chi) \) appear as constant terms of Eisenstein series of \( U(2,2) \)

(ii) An eigen modular form of \( U(2,2) \) produces a Galois representation

(iii) \( \text{Sel}_{pn}(E/F) \)

\[
= \frac{1}{2} \text{ extensions} \\
0 \to E[p^n] \to * \to \mathbb{Z}/p^n\mathbb{Z} \to 0
\]
III Non-commutative Iwasawa theory

$E/Q$ elliptic curve

good ordinary reduction at $p$

$F_n = \mathbb{Q}(E[p^n])$, $F_\infty = \bigcup_n F_n$

$G = \text{Gal}(F_\infty/\mathbb{Q})$ non-commutative

$\mathbb{Z}_p[[G]] = \lim_{\leftarrow} \mathbb{Z}_p[\text{Gal}(F_n/\mathbb{Q})]$ non-commutative
Arithmetic side

\[ X = \text{Hom} \left( \lim_{n} \text{Sel}(E/F_n), \mathbb{Q}_p/\mathbb{Z}_p \right) \]

\[ \mathbb{Z}_p[[G]]\text{-module} \]

A big problem was

Zeta side

Where does
the p-adic L-function
Arithmetic side

\[ X = \text{Hom} \left( \lim_{n} \text{Sel}(E/F_n), \mathbb{Q}_p/\mathbb{Z}_p \right) \]

\[ \mathbb{Z}_p[[G]] \text{-module} \]

A big problem was

Zeta side

Where does
Non-commutative rings are not good places to live.

For complex $L$-functions of the form of Euler product

$(\text{factor at } 2) \times (\text{factor at } 3) \times (\text{factor at } 5) \times \ldots$

the meaning of Euler product becomes unclear by the non-commutativity.

But

the non-commutativity of a ring $A$

vanishes under

$A^* = \text{GL}_1(A)$
Zeta can live in $K_1$ of a non-commutative ring

Example Selberg zeta function

$\Gamma \subset \text{SL}_2(\mathbb{R})$ discrete, co-compact

$Z_\Gamma (s) = \prod_{\gamma \in \Gamma/\mathcal{N}} (1 - N(\gamma)^{-s})^{-1} \in \mathbb{C}^x$

$\gamma: \text{prime}$

$\tilde{Z}_\Gamma (s) = \prod_{\gamma \in \Gamma/\mathcal{N}} (1 - \gamma N(\gamma)^{-s})^{-1} \in K_1(L^2(\Gamma))$

$L^2(\Gamma) = \left\{ \sum_{\gamma \in \Gamma} a_{\gamma} \gamma \left| \sum_{\gamma} |a_{\gamma}| < \infty \right. \right\}$
Zeta can live in $K_1$ of a non-commutative ring

Example Selberg zeta function

$\Gamma \subset \text{SL}_2(\mathbb{R})$ discrete, co-compact

\[
Z_{\Gamma}(s) = \prod_{\gamma \in \Gamma / \sim} (1 - N(\gamma)^{-s})^{-1} \in \mathbb{C}^\times
\]

$\gamma$ prime

$\tilde{\Sigma}_{\Gamma}(s) = \prod_{\gamma \in \Gamma / \sim} (1 - \gamma N(\gamma)^{-s})^{-1} \in K_1(L^2(\Gamma))$

$\gamma$ prime

$L^2(\Gamma) = \{ \sum_{\gamma \in \Gamma} a_\gamma \gamma \mid \sum_{\gamma} |a_\gamma|^2 < \infty \}$
Conjecture (Coates, Fukaya, K., Susarla, Venjakob)

There exists

\[ L_p(E, F_{\infty}/\mathbb{Q}) \in K_2(O[[G]]_{S^r}) \]

which \( p \)-adically interpolates

\[ L(E, 1, p) \]

period

\[ \text{finite image} \]

Here \( O = \hat{\mathbb{Z}}_p \)

\[ S = \{ f \in O[[G]] | O[[G]]/O[[G]] f \text{ is f.g. as an } O[[H]] \text{-module} \} \]

\[ (H = \text{Gal}(F_{\infty}/\mathbb{Q}^{nc}) \leq G) \]

\[ S^r = \bigcup S_{p^n} \]
Main conjecture (Coates et al.)

\[
X_0 = \mathcal{O}[[G]] \otimes_{\mathbb{Z}_p[[G]]} X \text{ is } S^*\text{-torsion}
\]

and

\[
K_1(\mathcal{O}[[G]]_{S^*}) \xrightarrow{\partial} K_0(\mathcal{O}[[G]], \Omega^1 \mathcal{O}[[G]]_{S^*}) \xrightarrow{\omega} K_0(\Omega^1 \mathcal{O}[[G]], \Omega^2 \mathcal{O}[[G]]_{S^*}) \xrightarrow{\omega} L_p(E, \mathcal{F}_n/\mathbb{G}) \implies [X_0]
\]

(compatible with Conjectures of Burns-Flach, Huber-Kings)
Recall:

ζ(-1), ζ(-3), ζ(-5), ...

→ Classical Iwasawa theory

unified

Dream:

commutative Iwasawa theories of various zeta functions

→ Unified non-commutative 

unified

Iwasawa theory
(2) Kummer (middle 19th C)

(2A) Kummer's congruence

\[ p: \text{prime}, \ r, r' \in \mathbb{Z}_{>0} \]

\[ r \equiv r' \not\equiv 1 \mod{p-1} \]

\[ \Rightarrow \ z(r) \equiv z(r') \mod{p} \]

---

generalized as

\[ \Rightarrow \ \text{congruences } \mod{p^n}, \ n \geq 1 \]

\[ \Rightarrow \ \exists \ \text{p-adic Riemann zeta function} \]
\( 5(-1), 5(-3), 5(-5), \ldots \)

\[ \rightarrow \text{Classical Iwasawa theory} \]

unified

Dream:

- commutative Iwasawa theories of various zeta functions

\[ \rightarrow \text{Unified non-commutative} \]

unified

Iwasawa theory
\[ r ! \equiv 1 \mod p \Rightarrow \]

\[ \zeta(r) \equiv \zeta(r') \mod p \]

generalized as

\[ \Rightarrow \text{ congruences mod } p^n, \quad n \geq 1 \]

\[ \Rightarrow \exists \text{ } p\text{-adic Riemann zeta function} \]
\[ \text{(Kubota-Leopoldt 20C)} \]
\[ \text{which } p\text{-adically interpolates} \]
\[ \zeta(r), \quad r \in \mathbb{Z}_{\leq 0} \]
(2B) Kummer's criterion

\[ p : \text{prime} \]
\[ p \mid 5(r) \quad \exists r \in \mathbb{Z}_{\leq 0} \quad r : \text{odd} \]
if and only if
\[ \mathbb{Z}/p\mathbb{Z} \subset \text{Cl}(\mathbb{Q}(5_p)) \]
(i.e. \( p \mid \# \text{Cl}(\mathbb{Q}(5_p)) \))

Here \( 5_n = \exp\left(\frac{2\pi i}{n}\right) \)
primitive \( n \)-th root of 1

Example \( p \mid \# \text{Cl}(\mathbb{Q}(5_p)) \) for \( p = 691 \)
- ideal class group is a bitter group which makes number theory harder

- ideal class group is a sweet group which has sweet relations with zeta values