THE WORK OF WENDELIN WERNER

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Since 2001, there has been an explosion of interest in and applications of the SLE approach. To discuss this, we first describe SLE.

For, say, a Jordan domain D in the plane with distinct a, b on its boundary ∂D, and $\kappa > 0$, (chordal) SLE_κ is a certain random continuous path in the closure $\bar{D}$ from a to b. When $\kappa \leq 4$, SLE_κ is a simple path that only touches ∂D at a and b. Loewner, in work dating back to the 20's, studied the evolution from a to b of nonrandom curves in terms of a continuous real-valued "driving function," $U(t)$. SLE_κ corresponds to $U(t) = B(\kappa t)$ where B is one-dimensional Brownian motion. When $\kappa > 4$, SLE_κ is non-simple and for $\kappa \geq 8$ it becomes plane-filling.

Many SLE-based results were motivated by (nonrigorous) results in Statistical Physics about $d = 2$ critical phenomena. Physical critical points happen at specific values of parameters, such as where the vapor pressure curve ends in a liquid/gas system. Critical systems have remarkable properties, such as microscopic random fluctuations manifesting themselves macroscopically. Also many quantities exhibit power law behavior, with the non-integer powers, known as *critical exponents*, believed to satisfy "universality", e.g., microscopically distinct models with the same exponent values at their respective critical points. Critical systems in $d = 2$ have another remarkable property, at the heart of both the SLE approach and its Physics predecessors — conformal invariance on the macroscopic scale.

Many SLE-based results are proofs of exponent values derived earlier by nonrigorous arguments such as those of "Conformal Field Theory" (CFT), which dates back to Polyakov and many others in the 70's, 80's, 90's; others were brand new. I'll discuss a few below, but most exciting is that the SLE approach is conceptually quite complementary to CFT. Werner has emphasized that complementary relationship including a focus on the "restriction property", as in his paper on the conformally invariant measure on self-avoiding loops. That is part of a burgeoning interest in extending the original focus on random curves to random *loops*, still with conformal invariance — both for percolation scaling limits and in the general context of Conformal Loop Ensembles, currently studied by Scott Sheffield and Werner.

Here are more examples of the results of recent years.

Let $W(t)$ be a planar Brownian motion. The complement in the plane of the curve segment $W([0,t])$ is a countable union of open sets, one of which is infinite; the boundary of that infinite component is the Brownian *frontier*. Using deep relations with SLE_6, Lawler, Schramm and Werner proved a celebrated 1982 conjecture of Mandelbrot:

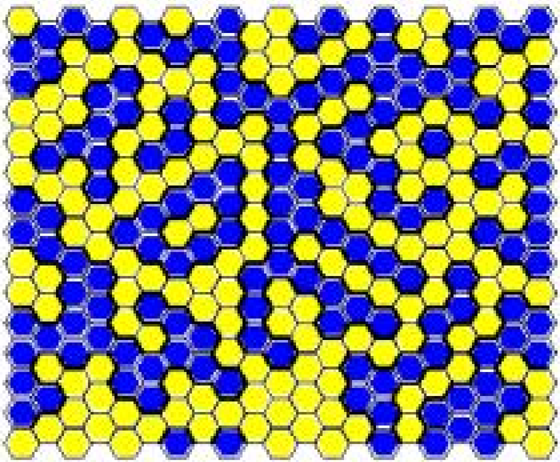
Theorem 2. *The Hausdorff dimension of the Brownian frontier is* $4/3$.

Another set of results are stated informally in the next theorem. They concern loop-erased random walks and related objects on lattices. Unlike the percolation case discussed next, these results about continuum scaling limits, in which the lattice scale shrinks to zero, are not restricted to a particular lattice.

Theorem 3. *For D (say) a Jordan domain, the scaling limits in D of loop-erased random walk, the uniformly random spanning tree and the related lattice-filling curve are, respectively (radial) SLE_2, a continuum "SLE_2-based tree" and the plane-filling (chordal) SLE_8.*

Scaling limits of lattice models are among the most interesting results. They require the combination of techniques from three different areas: conformal geometry (as in the classical [non-random] Loewner evolutions), stochastic analysis (since for SLE the driving function is random), and the probability theory of lattice models (e.g., random walks, or percolation or Ising models or ...). The work of Werner combines all three ingredients admirably well.

Another example of how these three areas interact is scaling limits of percolation. Physicists knew (nonrigorously) exponent values and some geometrical information — i.e., the formulas of Cardy for scaling limits of crossing probabilities. But there was no understanding of the scaling limit geometry of objects like cluster "interfaces."



Cluster interfaces indicated by heavy lines.

Then Schramm argued that the limit of the particular "exploration path" interface should be SLE_6. Next, for the triangular lattice, Smirnov proved that (A) crossing probabilities do converge to the conformally invariant Cardy formulas, sketched how that could lead to (B) convergence of the exploration path to SLE_6 and argued that one should be able to obtain (C) a "full scaling limit" for the family of "interface loops" of all clusters. Then, Smirnov and Werner proved certain percolation exponents, using exploration path convergence (B), while Lawler, Schramm and Werner combined the full scaling limit (C) with percolation arguments to prove another exponent value (see below).

Convergence in (B) and (C) can be proved by using lattice percolation machinery including results of Kesten, Sidoravicius and Zhang about six-fold crossings of annuli and of Aizenman, Duplantier and Aharony about narrow "fjords." Then the results of Werner and coauthors apply and prove a prediction of den Nijs and Nienhuis:

Theorem 4. *In critical site percolation on the triangular lattice, the probability,* $\mathrm{P}(R)$*, that the cluster of the origin has diameter greater than* R *is*

$$\mathrm{P}(R) = R^{-5/48+o(1)} \quad \text{as} \quad R \to \infty \quad .$$

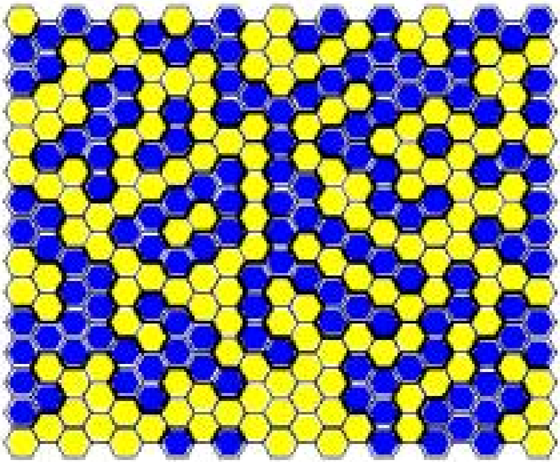
Traditionally, much of Probability Theory has focussed on continuum objects such as Brownian Motion and Stochastic Calculus, with SLE as the latest in the pantheon. Those of us raised in Statistical Mechanics sometimes regard lattice models as more "real." But this is a narrow view. Continuum models have extra properties, like conformal invariance for $d = 2$, that relate Probability Theory to other areas of Mathematics — relations that are of increasing importance. Even for the original lattice models, features like critical exponents and universality require a deep analysis of the continuum models in the scaling limit. Thanks to the work of Wendelin Werner, his collaborators, and others, it is fair to say that now

we are all "continuistas."

It is my pleasure to report on some of Wendelin Werner's research that has led to his Fields Medal at ICM–2006. There are aspects of Werner's work that add to my pleasure. One is that he trained as a probabilist, receiving his Ph.D. in 1993 under the supervision of Le Gall in Paris. His dissertation concerned planar Brownian Motion, which plays a major role in his later work as well. Until now, Probability Theory had not been represented among Fields Medals and so I am very pleased to be here to witness a change in that history.

Werner's work, with collaborators such as Greg Lawler, Oded Schramm and Stas Smirnov, involves applications of Probablity and Conformal Mapping Theory to fundamental problems in Statistical Physics. A second pleasure is that this, together with other work of recent years, represents a watershed in the interaction beteen Mathematics and Physics. Mathematicians such as Werner not only provide rigorous proofs of already existing claims in the Physics literature, but also new conceptual understanding of basic phenomena — in this case, a direct geometric picture of the intrinsically random structure of physical systems at their critical points (at least in dimension $d = 2$).

One important example is percolation.



Portion of a percolation configuration on the triangular lattice. Hexagons represent sites and are assigned one of two colors. In critical percolation, colors are assigned randomly with equal probability.

Permit me a remark as director of the Courant Institute. We have a scientific viewpoint, as did our predecessor institute in Göttingen — that a goal should be the elimination of artificial distinctions between the Mathematical Sciences and their applications — Wendelin Werner's work lives up to that philosophy.

A third pleasure concerns the collaborative nature of much of Werner's work. Beautiful, productive mathematics can result from many different workstyles. But the highly interactive style, of which Werner, together with his collaborators, is a leading exemplar, appeals as simultaneously good for the soul while leading to work stronger than the sum of its parts. It bodes well to see a Fields Medal awarded for this style of work.

The area of Probability Theory which most strongly interacts with Statistical Physics is that involving stochastic processes with non-trivial spatial structure. This area has long combined interesting applications with first-class Mathematics. Recent developments have raised the status of the best work from "merely" first-class to outstanding. Let me mention two pieces of Werner's work from 1998-2000. These are of intrinsic significance and also were among the precursors to breakthroughs about to happen in the understanding of $d = 2$ critical systems with (natural) conformal invariance. (Other precursors include Aizenman's path approach to scaling limits and Kenyon's work on loop-erased walks.)

The first is a 1998 paper of Bálint Tóth and Werner that constructed a continuum version of Tóth's earlier lattice "true self-repelling walk." This involved a beautiful mathematical structure (extending a nearly forgotten construction done decades earlier by Arratia) of coalescing and "reflecting" one-dimensional Brownian paths, running forward and backward in time and filling up all of space-time. A (random) plane-filling curve in this structure is analogous to one in scaling limits of uniformly random spanning trees and was one of Schramm's motivations in his 2000 paper about SLE. SLE is an acronym for Stochastic/Schramm-Loewner Evolution; more about SLE shortly.

The second work consists of two papers with Lawler in 1999 and 2000 involving planar Brownian intersection exponents. In the second of these, it was shown that the same set of exponents must occur providing only that certain locality and conformal invariance properties are valid. This key idea, combined with the introduction of SLE for the analysis of $d = 2$ critical phenomena, led to a remarkable series of three papers in 2001-2002 by Lawler, Schramm and Werner which yielded a whole series of intersection exponents.

E.g., let $W^1(t), W^2(t), \ldots$ be independent planar Brownian motions from distinct points at $t = 0$. Then the probability that the segments $W^1([0,t])$, ..., $W^n([0,t])$ are disjoint is $t^{-\zeta_n + o(1)}$ as $t \to \infty$ for constants ζ_n.

Theorem 1. *The intersection exponents ζ_n, for $n \geq 2$, are given by*

$$\zeta_n = \frac{4n^2-1}{24} \quad .$$

This was conjectured by Duplantier and Kwon and derived nonrigorously by Duplantier using $d = 2$ quantum gravity. Despite the simplicity of the formula, prior to SLE, its proof by conventional probabilistic techniques had been totally out of reach.