Moduli Spaces
from a Topological viewpoint

discussing work of primarily

Søren Galatius
Ib Madsen
Ulrike Tillmann
Michael Weiss

— in various combinations.
The collection $MT(d) = \{\text{Th}(U_{d,n})\}_n$ is a “spectrum”. Its associated infinite loop space is

$$\Omega^\infty MT(d) = \lim_{n \to \infty} \Omega^{n+d} \text{Th}(U_{d,n}).$$

**Theorem (Madsen-Weiss)** The map $\alpha_g : \mathbb{Z} \times M_g^{\text{top}} \to \Omega^\infty MT(2)$ induces an isomorphism on integral cohomology in degrees $k < \frac{g-2}{2}$.

**Corollary (i)** $\mathbb{Z} \times H^*(BT_\infty) \cong H^*(\Omega^\infty MT(2))$

**(ii)** $H^*(\Omega^\infty MT(2)) \otimes \mathbb{Q} \cong \text{Symm}_\mathbb{Q}(\tilde{H}^*(\text{Grass}_2(\mathbb{R}^\infty))) = \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$. 
III Cobordism Categories \((d \geq 0)\)

Two closed, oriented \((d - 1)\)-manifolds \(M_0, M_1\) are cobordant if there exist a compact, oriented \(d\)-manifold \(W^d\) with

\[
\partial W^d = M_1 \sqcup (-M_0) \quad (-M_0 = M_0 \text{ with opposite orientation}).
\]

\(\Omega_{d-1}^{SO}\) = set of equivalence classes (Pontryagin, Thom, Wall).

Category \(\mathcal{C}_d\): replace abstract manifolds by embedded ones, i.e.

object \(\mathcal{C}_d = \{(a, M^{d-1}) \mid a \in \mathbb{R}, M^{d-1} \subset \mathbb{R}^{d-1 + \infty} \text{ closed, oriented}\}
\]

\[= \mathbb{R} \times \bigsqcup B \text{Diff}(M)\] (union over diffeomorphism classes).

\(\text{Mor}((a_0, M_0), (a_1, M_1))\)

\[= \{W^d \subset [a_0, a_1] \times \mathbb{R}^{d-1 + \infty} \mid \partial W = a_1 \times M_1 \sqcup a_0 \times (-M_0) + \text{collar}\}
\]

\[= \mathbb{R} \times \mathbb{R}_+ \times \bigsqcup B \text{Diff}(W^d; \partial W^d).\]
Composition in $\mathcal{C}_d = \text{union of submanifolds}$.

To each (small) category $\mathcal{C}$ one can associate a classifying space $B\mathcal{C}$,

$B : \text{Categories} \rightarrow \text{Spaces}$

- in $\mathcal{C} \leftrightarrow 1$-simplex in $B\mathcal{C}$,
- in $\mathcal{C} \leftrightarrow 2$-simplex in $B\mathcal{C}$, etc.
Examples

(1) \( \text{ob} \mathcal{G} = \{ * \}, \text{mor} \mathcal{G} = G \) (group). Then \( B\mathcal{G} = BG \), the classifying space for principal \( G \)-bundles.

(2) For the cobordism category \( \mathcal{C}_d \), \( \pi_0 B\mathcal{C}_d = \Omega^\infty_{d-1} \).

Theorem A (GMTW) For all \( d \geq 0 \), \( \Omega B\mathcal{C}_d \simeq \Omega^\infty MT(d) \).

The proof uses geometric interpretation of the homotopy groups \( \pi_0 \Omega B\mathcal{C}_d \) and \( \pi_0 \Omega^\infty MT(d) \) via transversality and submersion theory.

- Inspiration to consider \( \mathcal{C}_d \) comes from field theories. Segal's category of Riemann surfaces with parametrized boundaries has the same classifying space as \( \mathcal{C}_2 \).

- Conformal field theory is a functor \( F : \mathcal{C}_d \to \{ \text{Hilbert spaces} \} \).
Theorem A potentially gives information about diffeomorphism groups and moduli spaces:

Each morphism \( W^d : (a_0, M_0^{d-1}) \rightarrow (a_1, M_1^{d-1}) \) in \( \mathcal{C}_d \) defines a 1-simplex (a path) in \( B\mathcal{C}_d \). The endpoints lie in the component determined by \([M_0] = [M_1]\) in \( \Omega_{d-1}^{SO} \). Connecting these points to a base point gives

\[
\gamma: \text{Mor}_{\mathcal{C}_d}((a_0, M_0), (a_1, M_1)) \rightarrow \Omega B\mathcal{C}_d.
\]

The morphism space is a union of \( B\text{Diff}(W^d; \partial) \). We seek "large" \( W^d \), so that

\[
\gamma: B\text{Diff}(W^d; \partial) \rightarrow \Omega B\mathcal{C}_d
\]

induces cohomology isomorphism onto a component in a range of dimensions.

For \( d = 2 \) Harer's stability theorem provides such a manifold. For \( d > 2 \) nothing similar is known.
Let $F_{g,b}$ be a surface of genus $g$ with $\partial F_{g,b} = \bigsqcup S^1$.

$\text{Diff}(F_{g,b}; \partial) = \text{orient. presv. diffeos that fixes } \partial F_{g,b} \text{ pointwise}$. The components are contractible, so $B \text{Diff}(F_{g,b}; \partial) \to B\Gamma_{g,b}$ is a homotopy equivalence, where $\Gamma_{g,b} = \pi_0 \text{Diff}(F_{g,b}; \partial)$. Consider the maps

$$
\text{Diff}(F_{g,b-1}; \partial) \hookrightarrow \text{Diff}(F_{g,b}; \partial) \twoheadrightarrow \text{Diff}(F_{g+1,b}; \partial)
$$

### Theorem (Harer 1986)

For $2k < g - 2$ there are isomorphisms

$$
H^k(B\Gamma_{g,b-1}) \xrightarrow{\cong} H^k(B\Gamma_{g,b}) \xleftarrow{\cong} H^k(B\Gamma_{g+1,b}).
$$
Positive boundary subcategory $\mathcal{C}_d^0 \subset \mathcal{C}_d$: same objects, less morphisms:

\[ \mathcal{C}_d^0 \]

**Theorem B** (GMTW) $B\mathcal{C}_d^0 \simeq B\mathcal{C}_d$ ($d \neq 1$).

- Theorem A + Theorem B $\implies \Omega B\mathcal{C}_d^0 \simeq \Omega^\infty MT(2)$.

A new proof of the Madsen-Weiss Theorem now follows from:

**Theorem** There is a homology isomorphism $\mathbb{Z} \times B\Gamma_{\infty,0} \to \Omega B\mathcal{C}_2^0$. 
One constructs a space $\mathcal{X}$ with a map $\pi : \mathcal{X} \to B\Theta^3_2$ such that

(i) $\pi^{-1}(\sqcup S^1) \simeq BT_{\infty,\rho+1}$

(ii) $\mathcal{X}$ is contractible (deforms to a point).

Harer + (i) $\Rightarrow$ all fibers of $\pi$ are homology equivalent,
McDuff-Segal + (ii) $\Rightarrow$ $\pi^{-1}(\sqcup S^1) \to \Omega B\Theta^3_2$ homology isomorphism.

**Remark** There are similar theorems for Riemann surfaces with marked points, but stably one just gets a decomposition of cohomology

$$H^*(BT^3_{\infty, \rho}) = H^*(BT_{\infty, \rho+\delta}) \oplus \mathbb{Z}[\lambda_1, \ldots, \lambda_\rho]; \quad \lambda_i = c_2(L_i).$$
IV Auxiliary results

Variations of $\mathcal{C}_d$: One can consider spin manifolds, unoriented manifolds, manifolds with a map to a background space $X$, graphs, etc.

- Versions of Theorems A and B remain valid, but stability theorems must be proved in each case ($\Rightarrow d = 2$). This is done by T. Bauer and N. Wahl for spin and unoriented manifolds, respectively.

\[ \mathcal{M}_g^{\text{top}}(X) = \{(\Sigma, f) \mid \Sigma \subset \mathbb{R}^{\infty+2}, \text{ genus } g, f : \Sigma \to X \text{ cont. map} \} \]

"Topological Gromov-Witten space"
**Theorem** (Cohen-Madsen) If $X$ is simply connected, then

(i) $H^*(\mathcal{M}_g^{\text{top}}(X))$ is independent of $g$ for $2^* < g - 4$,  
(ii) $\mathbb{Z} \times H^*(\mathcal{M}_g^{\text{top}}(X)) \cong H^*(\Omega^\infty(\mathcal{M}T(2) \wedge X_+))$.

**Corollary** $\pi_0\mathcal{M}_g^{\text{top}}(X) = H_2(X)$, and for each component we have

$$H^*(\mathcal{M}_g^{\text{top}}(X)_c) \otimes \mathbb{Q} = \text{Free Comm. Alg. } (H^{*+2}(\mathbb{C}P^\infty \times X)[-2] \otimes \mathbb{Q}).$$

(2) $\text{Aut}_n = \{\text{automorphisms of free group on } n \text{ letters}\}$,

$\Omega^n S^n = \{\text{maps } f : S^n \to S^n \text{ with } f(\infty) = \infty\}$.

**Theorem** (Galatius) There is a homology isomorphism

$$\gamma : \mathbb{Z} \times B\text{Aut}_\infty \to \Omega^\infty S^\infty.$$

**Corollary** For $k > 0$, $H^k(B\text{Aut}_\infty) \otimes \mathbb{Q} = 0$. 
- Differentiable orientable, connected surfaces are classified (up to diffeomorphism) by their genus: The moduli space is the set of non-negative integers.

\[ F_g : \]

- Riemann Surfaces (= complex structures on \( F_g \)), in contrast, depend on \( 6g - 6 \) real parameters: The moduli space \( \mathcal{M}_g \) is a variety of real dimension \( 6g - 6 \). \( \mathcal{M}_0 = \{ * \}, \mathcal{M}_1 = \mathbb{R}^2 \): Take \( g \geq 2 \) from now on.
Steps in Galatius' proof:

- Replace $\mathcal{C}_d$ by cobordism category $\mathcal{G}$ of embedded graphs:

- Stability theorem of Hatcher-Vogtmann-Wahl $\Rightarrow$
  Homology isomorphism $\mathbb{Z} \times B \text{Aut}_\infty \to \Omega B\mathcal{G}^\partial$.

- Homotopy equivalence $B\mathcal{G}^\partial \simeq B\mathcal{G}$.

- Homology isomorphism $\Omega B\mathcal{G} \to \Omega^\infty S^\infty$. This uses "scanning", i.e. Gromov's theory of flexible sheaves + a hard calculation.
V On the Deligne-Mumford compactification $\overline{M}_g$

Study of $\overline{M}_g$ involved two maps

(a) $\vartheta : B\text{Diff}(F_g) \to \overline{M}_g$ \quad ($\overline{M}_g^{\text{top}} \simeq B\text{Diff}(F_g) \simeq BG_2$)

(b) $\alpha : B\text{Diff}(F_g) \to \Omega^\infty MT(2)$.

Eliashberg-Galatius have given analogous constructions for $\overline{M}_g$:

- Replace $B\text{Diff}(F_g)$ by classifying space $BLF_g$ of Lefschetz fibrations.
- Replace $\Omega^\infty MT(2)$ by similar space $\Omega^\infty MLF(2)$.

(a) $\overline{\vartheta} : BLF_g \to \overline{M}_g$, $\overline{\vartheta}^* : H^*(\overline{M}_g) \otimes \mathbb{Q} \xrightarrow{\cong} H^*(BLF_g) \otimes \mathbb{Q}$

(b) $\overline{\alpha} : BLF_g \to \Omega^\infty MLF(2)$.

$H^*(\Omega^\infty MLF(2)) \otimes \mathbb{Q} = \mathbb{Q}[[\ldots, \kappa_i, \ldots]] \otimes \mathbb{Q}[[\ldots, \lambda_{ij}, \ldots]]$

$\deg(\lambda_{ij}) = 2i + 4j + 2$. 
A Lefschetz fibration is a proper smooth $\pi : E^{n+2} \to X^n$ such that

- Locally in $E$, $\pi(x_1, \ldots, x_{n-2}, z_1, z_2) = (x_1, \ldots, x_{n-2}, z_1, z_2)$
  $(x_i \in \mathbb{R}, \ z_i \in \mathbb{C})$

- Singular set $\Sigma^{n-2} \subset E^{n+2}$ submanifold with $C^2$ normal bundle $U$

- The restriction $\pi|_\Sigma$ is an immersion with $C$ normal bundle $L$

$\lambda_{ij} := (\pi|_\Sigma)_!(c_1(U)^i c_2(U)^j)$.
\( \bar{M}_g \) is a projective variety with fundamental class \([\bar{M}_g] \in H_{6g-6}(\bar{M}_g) \otimes \mathbb{Q} \).

**Problem** Calculate the image of \([\bar{M}_g]\) under

\[
(\bar{\alpha}_g)_+ \circ (\bar{\theta}_+)^{-1} : H_+(\bar{M}_g) \otimes \mathbb{Q} \rightarrow H_+(\Omega^\infty \text{MLF}(2) \otimes \mathbb{Q}).
\]

Similar question for the Gromov-Witten moduli space of pseudo holomorphic curves in a symplectic background.
Tautological ring:

\[ R^*(\mathcal{M}_g) = \text{Image}(\mathbb{Q}[\kappa_1, \kappa_2, \ldots]) \to H^*(\mathcal{M}_g) \otimes \mathbb{Q}). \]

**Faber’s conjecture:** \( R^*(\mathcal{M}_g) \) satisfies Poincaré duality in real degree \( 2g - 4 \),

(i) \( R^{2g-4}(\mathcal{M}_g) = \mathbb{Q}\langle \kappa_{g-2} \rangle \) (proved by Looijenga)

(ii) \( R^k(\mathcal{M}_g) \otimes R^{2g-4-k}(\mathcal{M}_g) \to R^{2g-4}(\mathcal{M}_g) \) perfect paring.

S. Morita proved that \( \kappa_1, \ldots, \kappa_{[g/3]} \) generates \( R^*(\mathcal{M}_g) \). The argument uses the action of \( \Gamma_g \) on the lower central series for \( \pi_1(F_g) + \text{Sp}_{2g}(\mathbb{Z}) \) invariance theory.
$\mathcal{M}^s_g(X)$ moduli space with $s$ marked points (or punctures) + map to the manifold $X$.

Restricting to neighborhoods of marked points gives a map:

$$\text{Res} : \mathcal{M}^s_g(X) \to (LX // \text{Diff}(S^1))^s.$$  

$LX =$ space of parameterized loops in $X$; $//$ indicates the Borel orbit space.

Divide marked points into incoming and outgoing ones:

$$[LX // \text{Diff}(S^1)]^{s_{\text{in}}} \leftrightarrow \mathcal{M}^s_g(X) \rightarrow (LX // \text{Diff}(S^1))^{s_{\text{out}}}$$

**D. Sullivan:** $H_\ast(\mathcal{M}^s_g(X)) \otimes \mathbb{R}$ acts as operations

$$H_\ast(LX // \text{Diff}(S^1)^{s_{\text{in}}}) \otimes \mathbb{R} \rightarrow H_\ast(LX // \text{Diff}(S^1)^{s_{\text{out}}}) \otimes \mathbb{R}.$$  

The action even extends to $H_\ast(\mathcal{M}^s_g(X); \mathbb{R})$.  

V On the Deligne-Mumford compactification $\overline{M}_g$

Our results on $M_g$ and $B\Gamma_g$ involve two basic maps, namely

(a) $\theta : B\text{Diff}(F_g) \to M_g$, \quad ($M_g^{\text{top}} \simeq B\text{Diff}(F_g) \simeq B\Gamma_g$)

(b) $\alpha : B\text{Diff}(F_g) \to \Omega^\infty MT(2)$.

$B\text{Diff}(F_g)$ classifies genus $g$ surface bundles. The space $\Omega^\infty MT(2)$ classifies "formal surface bundles": Triples $(f, L, \phi)$ where

- $f : E^{n+2} \to X^n$ smooth, proper; $L$ complex line bundle over $E$

- $\phi : T_E \oplus \mathbb{R} \xrightarrow{\cong} f^*TX \oplus L \oplus \mathbb{R}$.

An orient. surface bundle $\pi : E \to X$ induces a formal surface bundle with $f = \pi$, $L = T^\pi E$ and $\phi$ induced from the isomorphism $T_E \cong T^\pi E \oplus f^*TX$. 
Eliashberg-Galatius has given analogous constructions for $\mathcal{M}_g$:

A Lefschetz fibration is proper smooth $\pi : E^{n+2} \rightarrow X^n$ such that

- locally in $E$, $\pi(x_1, \ldots, x_{n-2}, z_1, z_2) = (x_1, \ldots, x_{n-2}, z_1 z_2)$ ($x_i \in \mathbb{R}, z_i \in \mathbb{C}$)
- singular set $\Sigma^{n-2} \subset E^{n+2}$ submanifold with $C^0$ normal bundle $U$
- The restriction $\pi|_E$ is an immersion with $C$ normal bundle $L$.

Lefschetz fibrations of genus $g$ are classified by a space $BLF_g$. There is the notion of a formal Lefschetz fibration. They are classified by a space $\Omega^MMLF(2)$, similar in spirit to $\Omega^MMT(2)$. There are maps

\[ (\Theta) \quad \Theta : BL_g \rightarrow \mathcal{M}_g; \quad \Theta : H^*(\mathcal{M}_g) \otimes \mathbb{Q} \xrightarrow{\cong} H^*(BLF_g) \otimes \mathbb{Q}. \]

\[ (\delta_g) \quad \delta : BLF_g \rightarrow \Omega^MMLF(2). \]

- $H^*(\Omega^MMLF_g) \otimes \mathbb{Q} = \mathbb{Q}[[\ldots, \kappa_i, \ldots]] \otimes \mathbb{Q}[[\ldots, \lambda_{ij}, \ldots]]$, \quad $\deg \lambda_{ij} = 2i + 4j + 2$.
- Given Lefschetz fibration $\pi : E \rightarrow X$, $\lambda_{ij} = (\pi|_E)_*(c_1(U)^i c_2(U)^j)$. 
Cohomology Groups:

- Associates to a space $X$ a sequence of abelian groups $H^k(X)$, $k=0,1,2,\ldots$
  - $f: X \to Y$ cont. map $\Rightarrow f^*: H^k(Y) \to H^k(X)$.
- If $X$ can be cont. deformed to $Y$ ($X \simeq Y$),
  then $H^k(X) \cong H^k(Y)$ for all $k$.
  (Partial converse due to J.H.C. Whitehead).
- Product: $H^k(X) \otimes H^l(X) \to H^{k+l}(X)$, so
  $H^*(X)$ is a graded ring.
p-dimensional \( H^p(X) \) is a quotient
of Abelian groups \( H^q(X) \), \( q=0,1,2, \ldots \).

- \( f : X \to Y \) cont. map \( \Rightarrow f^* : H^q(Y) \to H^q(X) \).
- If \( X \) can be cont. deformed to \( Y \) (XorY), then \( H^q(X) \cong H^q(Y) \) for all \( q \).
  (Partial converse due to J.H.C. Whitehead).
- Product: \( H^q(X) \otimes H^q(X) \to H^{q+q}(X) \), so \( H^{\bullet}(X) \) is a graded ring.
- \( H^q(X) \) measures "high-dimensional holes" in \( X \), e.g.
  \( H^q(F_3) = \mathbb{Z}_3^2 \).
\( \mathcal{M}_g \) is a meeting place for many areas of mathematics, e.g.

- **Complex Analysis**: Teichmüller Theory
- **Algebraic Geometry**: \( \mathcal{M}_g \) quasi-projective variety with compactification \( \overline{\mathcal{M}}_g \) (Deligne-Mumford), Mumford's conjecture.
- **Geometry and Physics**: Pseudo-holomorphic curves in symplectic background; conformal field theories; Intersection theory in \( \mathcal{M}_g \).
- **Group theory**: Mapping Class Group.
1 Moduli space and Mapping Class Group

$\mathcal{S}_c(TF_g) = \{\text{complex structures on tangent bundle } TF_g\}.$

$\xymatrix{ TF_g \ar[r]^J \ar@/_/[rd]_\pi & TF_g \ar@/_/[ld]_\pi \\
& F_g }$ 

$J^2 = -\text{id}; \{v, T v\} \text{ oriented basis for } v \neq 0.$

$\text{Diff}(F_g)$ topological group of orient. preserving diffeomorphisms.

$\text{Diff}_1(F_g) \triangleleft \text{Diff}(F_g)$ connected component of the identity.

$\Gamma_g = \text{Diff}(F_g)/\text{Diff}_1(F_g) = \pi_0 \text{Diff}(F_g)$ (Mapping Class Group)

Teichmüller space: $\mathcal{T}_g = \mathcal{S}_c(TF_g)/\text{Diff}_1(F_g) \cong \mathbb{R}^{6g-6}$

Moduli space: $\mathcal{M}_g = \mathcal{T}_g/\Gamma_g = \mathcal{S}_c(TF_g)/\text{Diff}(F_g)$

Distant goal: Compute the cohomology ring $H^*(\mathcal{M}_g)$. 
\mathcal{M}_g is singular (not a manifold) because \Gamma_g acts on \mathcal{D}_g with finite stabilizers. The mapping class group provides is a non-singular cover,

\begin{align*}
\theta_0 & : B\Gamma_g \to \mathcal{M}_g, \\
\theta_0^* & : H^*(\mathcal{M}_g) \otimes \mathbb{Q} \xrightarrow{\cong} H^*(B\Gamma_g) \otimes \mathbb{Q}.
\end{align*}

\{\Gamma_g\text{-covering space over }X\}/\text{Isom.} = \{\text{maps } X \to B\Gamma_g\}/\text{Homotopy}.

J. Harer (1986): \(H^k(B\Gamma_g)\) independent of \(g\) when \(g > 2k + 2\).

Stable cohomology: \(H^k(B\Gamma_\infty) := H^k(B\Gamma_g), \ g > 2k + 2\).

**Mumford's Conjecture:** The stable rational cohomology ring is

\[H^*(B\Gamma_\infty) \otimes \mathbb{Q} = \mathbb{Q}[\kappa_1, \kappa_2, \ldots], \quad \deg \kappa_i = 2i.\]

We need a different model for \(B\Gamma_g\):

\[\mathcal{M}_g^{CP}(n) = \{\text{orient. surfaces } \Sigma \subset \mathbb{R}^{n+2}, \ \Sigma \text{ diffeo. to } F_g\}\]

\[= \text{Emb}(F_g, \mathbb{R}^{n+2})/\text{Diff}(F_g) \quad \text{(orbit space)}.\]
\text{Emb}(F_g, \mathbb{R}^{n+2}) \text{ is the space of smooth embeddings with the Whitney topology of uniform convergence of all derivatives.}

\mathcal{M}_g^{\text{top}}(n) \subset \mathcal{M}_g^{\text{top}}(n+1) \subset \cdots; \quad \mathcal{M}_g^{\text{top}} = \bigcup \mathcal{M}_g^{\text{top}}(n).

\Sigma \subset \mathbb{R}^{n+2} \Rightarrow \text{inner product on } T_\Sigma \Sigma + \text{orient.} \Rightarrow \text{complex structure.}

This implies a map

\theta: \mathcal{M}_g^{\text{top}} \to \mathcal{M}_g.

The space \text{Emb}(F_g, \mathbb{R}^{\infty+2}) \text{ is contractible (Whitney), and}

\text{Emb}(F_g, \mathbb{R}^{\infty+2}) \times \text{Diff}(F_g) F_g \xrightarrow{\pi} \mathcal{M}_g^{\text{top}}

is the universal smooth \(F_g\) bundle:

\{F_g\text{-bundles over } X\}/\text{Isomorphism} = \{\text{maps } X \to \mathcal{M}_g^{\text{top}}\}/\text{Homotopy.}
Summary  There are homotopy equivalences

(i) \( M^\text{top}_g \simeq B\text{Diff}(F_g) \) and \( B\text{Diff}(F_g) \simeq B\Gamma_g \) (Earle-Eells), and

(ii) \( \theta^*: H^*(M_g) \otimes \mathbb{Q} \xrightarrow{\cong} H^*(M^\text{top}_g) \otimes \mathbb{Q} \).

The classes \( \kappa_i \)

\( \pi: E^{m+2} \to X^m \) smooth \( F_g \)-bundle; \( T^\pi E \) tangents along the fibers

\( \pi_1: H^{k+2}(E) \to H^k(X) \) “integration along the fibers”

\( \pi_1(c_1(T^\pi E)^{i+1}) = \kappa_i \in H^{2i}(X) \Rightarrow \kappa_i \in H^{2i}(B\Gamma_g) \)
II  Pontryagin-Thom theory and $\mathcal{M}_g^{\text{top}}$

Let $E$ be an $n$-dim. orient. vector bundle over compact manifold $M^d$. Thom space: $\text{Th}(E) = \text{one point compactification of } E,$

Three important facts:

(1) The complement of $M$ in $\text{Th}(E)$ deforms to $\infty$.

(2) Thom isomorphism: $H^i(M) \cong H^{i+n}(\text{Th}(E))$.

(3) $M^d \subset \mathbb{R}^{n+d}$ orient. submanifold; $E \subset \mathbb{R}^{n+d}$ open normal tube $E \cong \text{Normal bundle of } M^d \subset \mathbb{R}^{n+d}$. 
Collapsing the complement of \( E \) in \( \mathbb{R}^{n+d} \) to one point gives

\[ c_M : S^{n+d} \to \text{Th}(E) \]  
(Pontryagin-Thom collapse map).

We can collect the tangential and normal structure of (3) by combining the collapse map \( c_M \) with the Grassmanians:

\[ \text{Grass}_d(\mathbb{R}^{n+d}) = \{ \text{d-dim. lin. subspace } V \subset \mathbb{R}^{n+d} + \text{orientation} \}, \]

\[ U_{d,n}^\perp = \{(u, V) \mid u \perp V\}, \text{ an } n\text{-dim. vector bundle on Grass}_d(\mathbb{R}^{n+d}). \]

For \( x \in M^d \), the tangent fiber \( T_x M \in \text{Grass}_d(\mathbb{R}^{n+d}) \), and each \( u \in E_x \) is orthogonal to \( T_x M \) so belongs to \( U_{d,n}^\perp \). This gives the map

\[ \alpha(M^d) : S^{n+d} \xrightarrow{c} \text{Th}(E) \to \text{Th}(U_{d,n}^\perp), \]

and hence an element of the \((n+d)\) fold loop space,

\[ \alpha(M^d) \in \Omega^{n+d} \text{Th}(U_{d,n}^\perp) = \text{Map}((S^{n+d}, \infty), (\text{Th}(U_{d,n}), \infty)). \]