Moduli Spaces from a Topological viewpoint

discussing work of primarily

Søren Galatius
Ib Madsen
Ulrike Tillmann
Michael Weiss

– in various combinations.

The collection $MT(d) = \{\mathrm{Th}(U_{d,n}^{\perp})\}_n$ is a "spectrum". Its associated infinite loop space is

$$\Omega^\infty MT(d) = \operatorname*{dir\,lim}_{n\to\infty} \Omega^{n+d}\,\mathrm{Th}(U_{d,n}^{\perp}).$$

Theorem (Madsen-Weiss) The map $\alpha_g : \mathbb{Z} \times \mathscr{M}_g^{\mathrm{top}} \to \Omega^\infty MT(2)$ induces isomorphism on integral cohomology in degrees $k < \frac{g-2}{2}$.

Corollary (i) $\quad \mathbb{Z} \times H^*(B\Gamma_\infty) \cong H^*(\Omega^\infty MT(2))$

(ii) $\quad H^*(\Omega^\infty MT(2)) \otimes \mathbb{Q} \cong \mathrm{Symm}_{\mathbb{Q}}(\tilde{H}^*(\mathrm{Grass}_2(\mathbb{R}^\infty))) = \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$.

III Cobordism Categories ($d \geq 0$)

Two closed, oriented $(d-1)$-manifolds M_0, M_1 are *cobordant* if there exist a compact, oriented d-manifold W^d with

$$\partial W^d = M_1 \sqcup (-M_0) \qquad (-M_0 = M_0 \text{ with opposite orientation}).$$

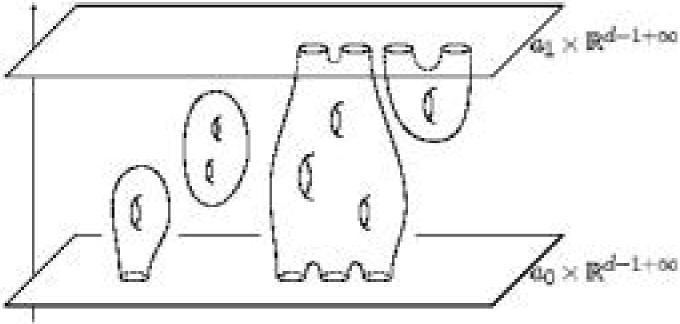
Ω^{SO}_{d-1} = set of equivalence classes (Pontryagin, Thom, Wall).

Category $\mathscr{C}_d$: replace abstract manifolds by embedded ones, i.e.

$$\begin{aligned}\text{object}\,\mathscr{C}_d &= \{(a, M^{d-1}) \mid a \in \mathbb{R},\ M^{d-1} \subset \mathbb{R}^{d-1+\infty} \text{ closed, oriented}\}\\ &= \mathbb{R} \times \bigsqcup B\,\mathrm{Diff}(M) \qquad \text{(union over diffeomorphism classes).}\end{aligned}$$

$$\begin{aligned}&\mathrm{Mor}((a_0, M_0), (a_1, M_1))\\ &\quad = \{W^d \subset [a_0, a_1] \times \mathbb{R}^{d-1+\infty} \mid \partial W = a_1 \times M_1 \sqcup a_0 \times (-M_0) + \text{collar}\}\\ &\quad = \mathbb{R} \times \mathbb{R}_+ \times \bigsqcup B\,\mathrm{Diff}(W^d; \partial W^d).\end{aligned}$$

$d = 2:$

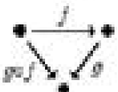


Composition in $\mathscr{C}_d$ = union of submanifolds.

To each (small) category $\mathscr{C}$ one can associate a classifying space $B\mathscr{C}$,

B : Categories $\longrightarrow$ Spaces

$\bullet \longrightarrow \bullet$ in $\mathscr{C} \longleftrightarrow$ 1-simplex in $B\mathscr{C}$,



in $\mathscr{C} \longleftrightarrow$ 2-simplex in $B\mathscr{C}$, etc.

Examples

(1) $\mathrm{ob}\,\mathscr{C} = \{*\}$, $\mathrm{mor}\,\mathscr{C} = G$ (group). Then $B\mathscr{C} = BG$, the classifying space for principal G-bundles.

(2) For the cobordism category $\mathscr{C}_d$, $\pi_0 B\mathscr{C}_d = \Omega_{d-1}^{SO}$.

Theorem A (GMTW) For all $d \geq 0$, $\Omega B\mathscr{C}_d \simeq \Omega^\infty MT(d)$.

The proof uses geometric interpretation of the homotopy groups $\pi_k \Omega B\mathscr{C}_d$ and $\pi_k \Omega^\infty MT(d)$ via transversality and submersion theory.

- Inspiration to consider $\mathscr{C}_d$ comes from field theories. Segal's category of Riemann surfaces with parametrized boundaries has the same classifying space as $\mathscr{C}_2$.

- Conformal field theory is a functor $F : \mathscr{C}_d \to \{\text{Hilbert spaces}\}$

Theorem A potentially gives information about diffeomorphism groups and moduli spaces:

Each morphism $W^d : (a_0, M_0^{d-1}) \to (a_1, M_1^{d-1})$ in $\mathscr{C}_d$ defines a 1-simplex (a path) in $B\mathscr{C}_d$. The endpoints lie in the component determined by $[M_0] = [M_1]$ in Ω^{SO}_{d-1}. Connecting these points to a base point gives

$$\gamma : \mathrm{Mor}_{\mathscr{C}_d}\big((a_0, M_0), (a_1, M_1)\big) \to \Omega B\mathscr{C}_d.$$

The morphism space is a union of $B\,\mathrm{Diff}(W^d; \partial)$. We seek "large" W^d, so that

$$\gamma : B\,\mathrm{Diff}(W^d; \partial) \to \Omega B\mathscr{C}_d$$

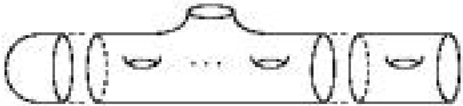
induces cohomology isomorphism onto a component in a range of dimensions.

For $d = 2$ Harer's stability theorem provides such a manifold.
For $d > 2$ nothing similar is known.

Let $F_{g,b}$ be a surface of genus g with $\partial F_{g,b} = \overset{b}{\sqcup} S^1$.

$\mathbf{Diff}(F_{g,b};\partial)$ = orient. presv. diffeos that fixes $\partial F_{g,b}$ pointwise. The components are contractible, so $B\,\mathrm{Diff}(F_{g,b};\partial) \to B\Gamma_{g,b}$ is a homotopy equivalence, where $\Gamma_{g,b} = \pi_0\,\mathrm{Diff}(F_{g,b};\partial)$. Consider the maps

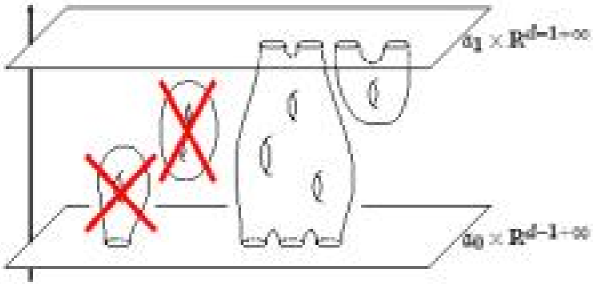
$$\mathbf{Diff}(F_{g,b-1};\partial) \longleftarrow \mathbf{Diff}(F_{g,b};\partial) \longrightarrow \mathbf{Diff}(F_{g+1,b};\partial)$$



Theorem (Harer 1986) For $2k < g-2$ there are isomorphisms

$$H^k(B\Gamma_{g,b-1}) \xrightarrow{\cong} H^k(B\Gamma_{g,b}) \xleftarrow{\cong} H^k(B\Gamma_{g+1,b}).$$

Positive boundary subcategory $\mathscr{C}_d^\partial \subset \mathscr{C}_d$: same objects, less morphisms:



Theorem B (GMTW) $B\mathscr{C}_d^\partial \simeq B\mathscr{C}_d \quad (d \neq 1)$.

- Theorem A + Theorem B $\quad \Rightarrow \quad \Omega B\mathscr{C}_2^\partial \simeq \Omega^\infty MT(2)$.

A new proof of the Madsen-Weiss Theorem now follows from:

Theorem There is a homology isomorphism $\mathbb{Z} \times B\Gamma_{\infty,b} \to \Omega B\mathscr{C}_2^\partial$.

One constructs a space $\mathscr{X}$ with a map $\pi : \mathscr{X} \to B\mathscr{C}_2^0$ such that

(i) $\pi^{-1}(\coprod^{c} S^1) \simeq B\Gamma_{\infty,c+1}$

(ii) $\mathscr{X}$ is contractible (deforms to a point).

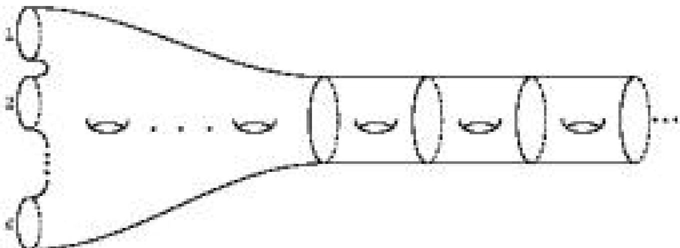
Harer + (i) $\Rightarrow$ all fibers of π are homology equivalent,

McDuff-Segal + (ii) $\Rightarrow \pi^{-1}(\coprod^{c} S^1) \to \Omega B\mathscr{C}_2^0$ homology isomorphism.

Remark. There are similar theorems for Riemann surfaces with marked points, but stably one just gets a decomposition of cohomology

$$H^*(B\Gamma^s_{\infty,g}) = H^*(B\Gamma_{\infty,g+s}) \otimes \mathbb{Z}[\lambda_1, \ldots, \lambda_s]; \quad \lambda_i = c_1(L_i).$$

$\pi^{-1}(\coprod^{c} S^1)$:



IV Auxiliary results

Variations of $\mathscr{C}_d$: One can consider spin manifolds, unoriented manifolds, manifolds with a map to a background space X, graphs, etc.

- Versions of Theorems A and B remain valid, but stability theorems must be proved in each case ($\Rightarrow d = 2$). This is done by T. Bauer and N. Wahl for spin and unoriented manifolds, respectively.

(1) $\mathscr{M}_g^{\mathrm{top}}(X) = \{(\Sigma, f) \mid \Sigma \subset \mathbb{R}^{\infty+2}$, genus g, $f : \Sigma \to X$ cont. map $\}$

"Topological Gromov-Witten space"

Theorem (Cohen-Madsen) If X is simply connected, then

(i) $H^*(\mathscr{M}_g^{\mathrm{top}}(X))$ is independent of g for $2* < g - 4$,

(ii) $\mathbb{Z} \times H^*(\mathscr{M}_g^{\mathrm{top}}(X)) \cong H^*(\Omega^\infty(MT(2) \wedge X_+))$.

Corollary $\pi_0 \mathscr{M}_g^{\mathrm{top}}(X) = H_2(X)$, and for each component we have

$$H^*(\mathscr{M}_g^{\mathrm{top}}(X)_c) \otimes \mathbb{Q} = \text{Free Comm. Alg.}\left(H^{*>2}(\mathbb{CP}^\infty \times X)[-2] \otimes \mathbb{Q}\right).$$

(2) $\mathrm{Aut}_n = \{\text{automorphisms of free group on } n \text{ letters}\}$,

$\Omega^n S^n = \{\text{maps } f : S^n \to S^n \text{ with } f(\infty) = \infty\}$.

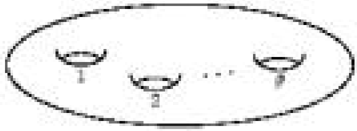
Theorem (Galatius) There is a homology isomorphism

$$\gamma : \mathbb{Z} \times B\,\mathrm{Aut}_\infty \to \Omega^\infty S^\infty.$$

Corollary For $k > 0$, $H^k(B\,\mathrm{Aut}_\infty) \otimes \mathbb{Q} = 0$.

- Differentiable orientable, connected surfaces are classified (up to diffeomorphism) by their genus: The moduli space is the set of non-negative integers.

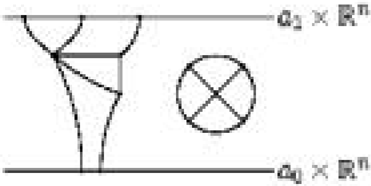
F_g :



- Riemann Surfaces (= complex structures on F_g), in contrast, depend on $6g - 6$ real parameters: The moduli space $\mathcal{M}_g$ is a variety of real dimension $6g - 6$. $\mathcal{M}_0 = \{*\}$, $\mathcal{M}_1 = \mathbb{R}^2$: Take $g \geq 2$ from now on.

Steps in Galatius' proof:

- Replace $\mathscr{C}_d$ by cobordism category $\mathscr{G}$ of embedded graphs:



(a morphism in $\mathscr{G}$)

- Stability theorem of Hatcher-Vogtmann-Wahl $\Rightarrow$ Homology isomorphism $\mathbb{Z} \times B\,\mathrm{Aut}_\infty \to \Omega B\mathscr{G}^\partial$.

- Homotopy equivalence $B\mathscr{G}^\partial \simeq B\mathscr{G}$.

- Homology isomorphism $\Omega B\mathscr{G} \to \Omega^\infty S^\infty$. This uses "scaning", i.e. Gromov's theory of flexible sheaves + a hard calculation.

V On the Deligne-Mumford compactification $\overline{\mathscr{M}}_g$

Study of $\mathscr{M}_g$ involved two maps

(a) $\theta : B\,\mathrm{Diff}(F_g) \to \mathscr{M}_g$ $\quad (\mathscr{M}_g^{\mathrm{top}} \simeq B\,\mathrm{Diff}(F_g) \simeq B\Gamma_g)$

(b) $\alpha : B\,\mathrm{Diff}(F_g) \to \Omega^\infty MT(2)$.

Eliashberg-Galatius have given analogous constructions for $\overline{\mathscr{M}}_g$:

- Replace $B\,\mathrm{Diff}(F_g)$ by classifying space $\mathrm{BL}F_g$ of Lefschetz fibrations.
- Replace $\Omega^\infty MT(2)$ by similar space $\Omega^\infty MLF(2)$.

($\bar{a}$) $\bar{\theta} : \mathrm{BL}F_g \to \overline{\mathscr{M}}_g$, $\bar{\theta}^* : H^*(\overline{\mathscr{M}}_g) \otimes \mathbb{Q} \xrightarrow{\cong} H^*(\mathrm{BL}F_g) \otimes \mathbb{Q}$

($\bar{b}$) $\bar{\alpha} : \mathrm{BL}F_g \to \Omega^\infty MLF(2)$.

- $H^*(\Omega^\infty MLF(2)) \otimes \mathbb{Q} = \mathbb{Q}[\ldots, \kappa_i, \ldots] \otimes \mathbb{Q}[\ldots, \lambda_{ij}, \ldots]$
  $\deg(\lambda_{ij}) = 2i + 4j + 2$.

A Lefschetz fibration is a proper smooth $\pi : E^{n+2} \to X^n$ such that

- Locally in E, $\pi(x, 1, \ldots, x_{n-2}, z_1, z_2) = (x_1, \ldots, x_{n-2}, z_1, z_2)$
  $(x_i \in \mathbb{R},\ z_i \in \mathbb{C})$
- Singular set $\Sigma^{n-2} \subset E^{n+2}$ submanifold with $\mathbb{C}^2$ normal bundle U
- The restriction $\pi|_\Sigma$ is an immersion with $\mathbb{C}$ normal bundle L.

$\lambda_{ij} := (\pi|_\Sigma)_! \left(c_1(U)^i c_2(U)^j\right)$.

$\overline{\mathscr{M}}_g$ is a projective variety with fundamental class $[\overline{\mathscr{M}}_g] \in H_{6g-6}(\overline{\mathscr{M}}_g) \otimes \mathbb{Q}$.

Problem Calculate the image of $[\overline{\mathscr{M}}_g]$ under

$$(\bar{\alpha}_g)_* \circ (\bar{\theta}_g^*)^{-1} : H_*(\overline{\mathscr{M}}_g) \otimes \mathbb{Q} \to H_*(\Omega^\infty MLF(2) \otimes \mathbb{Q}.$$

Similar question for the Gromov-Witten moduli space of pseudo holomorphic curves in a symplectic background.

Tautological ring:

$$R^*(\mathcal{M}_g) = \mathrm{Image}(\mathbb{Q}[\kappa_1, \kappa_2, \ldots]) \to H^*(\mathcal{M}_g) \otimes \mathbb{Q}).$$

Faber's conjecture: $R^*(\mathcal{M}_g)$ satisfies Poincaré duality in real degree $2g-4$,

(i) $R^{2g-4}(\mathcal{M}_g) = \mathbb{Q}\langle \kappa_{g-2} \rangle$ (proved by Looijenga)

(ii) $R^k(\mathcal{M}_g) \otimes R^{2g-4-k}(\mathcal{M}_g) \to R^{2g-4}(\mathcal{M}_g)$ perfect paring.

S. Morita proved that $\kappa_1, \ldots, \kappa_{[g/3]}$ generates $R^*(\mathcal{M}_g)$. The argument uses the action of Γ_g on the lower central series for $\pi_1(F_g) + \mathbf{Sp}_{2g}(\mathbb{Z})$ invariance theory.

$\mathscr{M}_g^s(X)$ moduli space with s marked points (or punctures) + map to the manifold X.

Restricting to neighborhoods of marked points gives a map:

$$\mathrm{Res} : \mathscr{M}_g^s(X) \to \left(LX//\,\mathrm{Diff}(S^1)\right)^s .$$

$LX =$ space of parameterized loops in X; // indicates the Borel orbit space.

Divide marked points into incoming and outgoing ones:

$$\left[LX//\,\mathrm{Diff}(S^1)\right]^{s_{in}} \longleftarrow \mathscr{M}_g^s(X) \longrightarrow \left(LX//\,\mathrm{Diff}(S^1)\right)^{s_{out}}$$

$$\left[LX//\,\mathrm{Diff}(S^1)\right]^{s_{in}} \overset{?}{\dashrightarrow} \mathscr{M}_g^s(X)$$

D. Sullivan: $H_*\left(\mathscr{M}_g^s(X)\right) \otimes \mathbb{R}$ acts as operations

$$H_*\left(LX//\,\mathrm{Diff}(S^1)^{s_{in}}\right) \otimes \mathbb{R} \to H_*\left(LX//\,\mathrm{Diff}(S^1)^{s_{out}}\right) \otimes \mathbb{R}.$$

The action even extends to $H_*(\overline{\mathscr{M}}_g(X); \mathbb{R})$.

V On the Deligne-Mumford compactification $\bar{\mathscr{M}}_g$

Our results on $\mathscr{M}_g$ and $B\Gamma_g$ involve two basic maps, namely

(a) $\theta : B\,\mathrm{Diff}(F_g) \to \mathscr{M}_g, \quad (\mathscr{M}_g^{\mathrm{top}} \simeq B\,\mathrm{Diff}(F_g) \simeq B\Gamma_g)$

(b) $\alpha : B\,\mathrm{Diff}(F_g) \to \Omega^\infty MT(2).$

$B\,\mathrm{Diff}(F_g)$ classifies genus g surface bundles. The space $\Omega^\infty MT(2)$ classifies "*formal* surface bundles": Triples (f, L, ϕ) where

- $f : E^{n+2} \to X^n$ smooth, proper; L complex line bundle over E
- $\phi : TE \oplus \mathbb{R} \xrightarrow{\cong} f^*TX \oplus L \oplus \mathbb{R}.$

An orient. surface bundle $\pi : E \to X$ induces a *formal* surface bundle with $f = \pi$, $L = T^\pi E$ and ϕ induced from the isomorphism $TE \cong T^\pi E \oplus f^*TX$.

Eliashberg-Galatius has given analogous constructions for $\bar{\mathcal{M}}_g$:

A Lefschetz fibration is proper smooth $\pi : E^{n+2} \to X^n$ such that

- locally in E, $\pi(x_1, \dots, x_{n-2}, z_1, z_2) = (x_1, \dots, x_{n-2}, z_1 z_2)$ ($x_i \in \mathbb{R}, z_i \in \mathbb{C}$)
- singular set $\Sigma^{n-2} \subset E^{n+2}$ submanifold with $\mathbb{C}^2$ normal bundle U
- The restriction $\pi|_\Sigma$ is an immersion with $\mathbb{C}$ normal bundle L.

Lefschetz fibrations of genus g are classified by a space $\mathrm{BL}F_g$. There is the notion of a formal Lefschetz fibration. They are classified by a space $\Omega^\infty MLF(2)$, similar in spirit to $\Omega^\infty MT(2)$. There are maps

$(\bar{3})$ $\bar{\partial} : \mathrm{BL}_g \to \bar{\mathcal{M}}_g$; $\bar{\partial}^* : H^*(\bar{\mathcal{M}}_g) \otimes \mathbb{Q} \xrightarrow{\cong} H^*(\mathrm{BL}F_g) \otimes \mathbb{Q}$.

$(\bar{5}_g)$ $\bar{\alpha} : \mathrm{BL}F_g \to \Omega^\infty MLF(2)$.

- $H^*(\Omega^\infty M\mathrm{FL}_g) \otimes \mathbb{Q} = \mathbb{Q}[\dots, \kappa_i, \dots] \otimes \mathbb{Q}[\dots, \lambda_{ij}, \dots]$, $\deg \lambda_{ij} = 2i + 4j + 2$.
- Given Lefschetz fibration $\pi : E \to X$, $\lambda_{ij} = (\pi|_\Sigma)_!(c_1(U)^i c_2(U)^j)$.

Cohomology Groups:

- Associates to a space X a sequence of abelian groups $H^k(X)$, $k=0,1,2,\cdots$
- $f: X \to Y$ cont. map $\Rightarrow f^*: H^k(Y) \to H^k(X)$.
- If X can be cont. deformed to Y ($X \simeq Y$), then $H^k(X) \cong H^k(Y)$ for all k.
  (Partial converse due to J.H.C. Whitehead)
- Product: $H^k(X) \otimes H^l(X) \longrightarrow H^{k+l}(X)$, so $H^*(X)$ is a graded ring.

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• Product: $H^k(X) \otimes H^l(X) \to H^{k+l}(X)$, so $H^*(X)$ is a graded ring.

• $H^k(X)$ measures "high-dimensional holes" in X, e.g.
$H^1(\Sigma_g) = \mathbb{Z}^{2g}$

$\mathscr{M}_g$ is a meeting place for many areas of mathematics, e.g.

- Complex Analysis: Teichmüller Theory
- Algebraic Geometry: $\mathscr{M}_g$ quasi-projective variety with compactification $\overline{\mathscr{M}}_g$ (Deligne-Mumford), Mumford's conjecture.
- Geometry and Physics: Pseudo-holomorphic curves in symplectic background; conformal field theories; Intersection theory in $\mathscr{M}_g$.
- Group theory: Mapping Class Group.

I Moduli space and Mapping Class Group

$\mathscr{S}_{\mathbb{C}}(TF_g) = \{$complex structures on tangent bundle $TF_g\}$.

$$\begin{array}{ccc} TF_g & \xrightarrow{J} & TF_g \\ & {\searrow}^{\pi} \quad {\swarrow}^{\pi} & \\ & F_g & \end{array} \quad ; \quad J^2 = -\mathrm{id}; \{v, Tv\} \text{ oriented basis for } v \neq 0.$$

$\mathrm{Diff}(F_g)$ topological group of orient. preserving diffeomorphisms.

$\mathrm{Diff}_1(F_g) \lhd \mathrm{Diff}(F_g)$ connected component of the identity.

$\Gamma_g = \mathrm{Diff}(F_g) / \mathrm{Diff}_1(F_g) = \pi_0 \mathrm{Diff}(F_g)$ (Mapping Class Group)

Teichmüller space: $\mathscr{T}_g = \mathscr{S}_{\mathbb{C}}(TF_g) / \mathrm{Diff}_1(F_g) \cong \mathbb{R}^{6g-6}$

Moduli space: $\mathscr{M}_g = \mathscr{T}_g / \Gamma_g = \mathscr{S}_{\mathbb{C}}(TF_g) \,/\, \mathrm{Diff}(F_g)$

Distant goal: Compute the cohomology ring $H^*(\mathscr{M}_g)$.

$\mathscr{M}_g$ is singular (not a manifold) because Γ_g acts on $\mathscr{T}_g$ with *finite* stabilizers. The mapping class group provides is a non-singular cover,

$$\theta_0 : B\Gamma_g \to \mathscr{M}_g, \quad \theta_0^* : H^*(\mathscr{M}_g) \otimes \mathbb{Q} \xrightarrow{\cong} H^*(B\Gamma_g) \otimes \mathbb{Q}.$$

$$\{\Gamma_g\text{-covering space over } X\}/\text{Isom.} = \{\text{maps } X \to B\Gamma_g\}/\text{Homotopy}.$$

J. Harer (1986): $H^k(B\Gamma_g)$ independent of g when $g > 2k+2$,

Stable cohomology: $H^k(B\Gamma_\infty) := H^k(B\Gamma_g)$, $g > 2k+2$.

Mumford's Conjecture: The stable rational cohomology ring is

$$H^*(B\Gamma_\infty) \otimes \mathbb{Q} = \mathbb{Q}[\kappa_1, \kappa_2, \ldots], \quad \deg \kappa_i = 2i.$$

We need a different model for $B\Gamma_g$:

$$\begin{aligned} \mathscr{M}_g^{\text{top}}(n) &= \{\text{orient. surfaces } \Sigma \subset \mathbb{R}^{n+2},\ \Sigma \text{ diffeo. to } F_g\} \\ &= \text{Emb}(F_g, \mathbb{R}^{n+2})/\text{Diff}(F_g) \qquad \text{(orbit space)}. \end{aligned}$$

$\mathbf{Emb}(F_g, \mathbb{R}^{n+2})$ is the space of smooth embeddings with the Whitney topology of uniform convergence of all derivatives.

$$\mathscr{M}_g^{\text{top}}(n) \subset \mathscr{M}_g^{\text{top}}(n+1) \subset \cdots; \quad \mathscr{M}_g^{\text{top}} = \bigcup \mathscr{M}_g^{\text{top}}(n).$$

$\Sigma \subset \mathbb{R}^{n+2} \Rightarrow$ inner product on $T_x\Sigma$ + orient. $\Rightarrow$ complex structure.

This implies a map

$$\theta : \mathscr{M}_g^{\text{top}} \to \mathscr{M}_g.$$

The space $\mathbf{Emb}(F_g, \mathbb{R}^{\infty+2})$ is contractible (Whitney), and

$$\text{Emb}(F_g, \mathbb{R}^{\infty+2}) \times_{\text{Diff}(F_g)} F_g \xrightarrow{\pi} \mathscr{M}_g^{\text{top}}$$

is the universal smooth F_g bundle:

$\{F_g\text{-bundles over } X\}/\text{Isomorphism} = \{\text{maps } X \to \mathscr{M}_g^{\text{top}}\}/\text{Homotopy}.$

Summary There are homotopy equivalences

(i) $\mathscr{M}_g^{\text{top}} \simeq B\,\text{Diff}(F_g)$ and $B\,\text{Diff}(F_g) \simeq B\Gamma_g$ (Earle-Eells), and

(ii) $\theta^* : H^*(\mathscr{M}_g) \otimes \mathbb{Q} \xrightarrow{\cong} H^*(\mathscr{M}_g^{\text{top}}) \otimes \mathbb{Q}$.

The classes κ_i

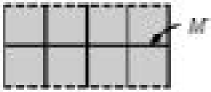
$\pi : E^{m+2} \to X^m$ smooth F_g-bundle; $T^\pi E$ tangents along the fibers

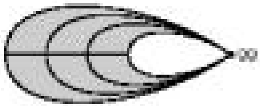
$\pi_! : H^{k+2}(E) \to H^k(X)$ "integration along the fibers"

$\pi_!(c_1(T^\pi E)^{i+1}) = \kappa_i \in H^{2i}(X) \;\Rightarrow\; \kappa_i \in H^{2i}(B\Gamma_g)$

II Pontryagin-Thom theory and $\mathscr{M}_g^{\text{top}}$

Let E be an n-dim. orient. vector bundle over compact manifold M^d.
Thom space: $\text{Th}(E) =$ one point compactification of E,

E : 

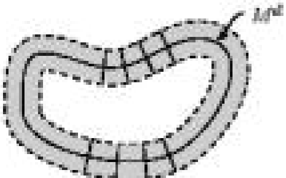
$\text{Th}(E)$: 

Three important facts:

(1) The complement of M in $\text{Th}(E)$ deforms to ∞.

(2) Thom isomorphism: $H^i(M) \cong H^{i+n}(\text{Th}(E))$.

(3) $M^d \subset \mathbb{R}^{n+d}$ orient. submanifold; $E \subset \mathbb{R}^{n+d}$ open normal tube

E : 

$E \cong$ Normal bundle of $M^d \subset \mathbb{R}^{n+d}$.

Collapsing the complement of E in $\mathbb{R}^{n+d}$ to one point gives

$$c_M : S^{n+d} \to \mathrm{Th}(E) \quad \text{(Pontryagin-Thom collapse map)}.$$

We can collect the tangential and normal structure of (3) by combining the collapse map c_M with the Grassmanians:

$\mathrm{Grass}_d(\mathbb{R}^{n+d}) = \{$ d-dim. lin. subspace $V \subset \mathbb{R}^{n+d}$ + orientation $\}$,
$U_{d,n}^{\perp} = \{(u, V) \mid u \perp V\}$, an n-dim. vector bundle on $\mathrm{Grass}_d(\mathbb{R}^{n+d})$.

For $x \in M^d$, the tangent fiber $T_x M \in \mathrm{Grass}_d(\mathbb{R}^{n+d})$, and each $u \in E_x$ is orthogonal to $T_x M$ so belongs to $U_{d,n}^{\perp}$. This gives the map

$$\alpha(M^d) : S^{n+d} \xrightarrow{c} \mathrm{Th}(E) \longrightarrow \mathrm{Th}(U_{d,n}^{\perp}),$$

and hence an element of the $(n+d)$ fold loop space,

$$\alpha(M^d) \in \Omega^{n+d}\mathrm{Th}(U_{d,n}^{\perp}) = \mathrm{Map}((S^{n+d}, \infty), (\mathrm{Th}(U_{d,n}^{\perp}), \infty)).$$