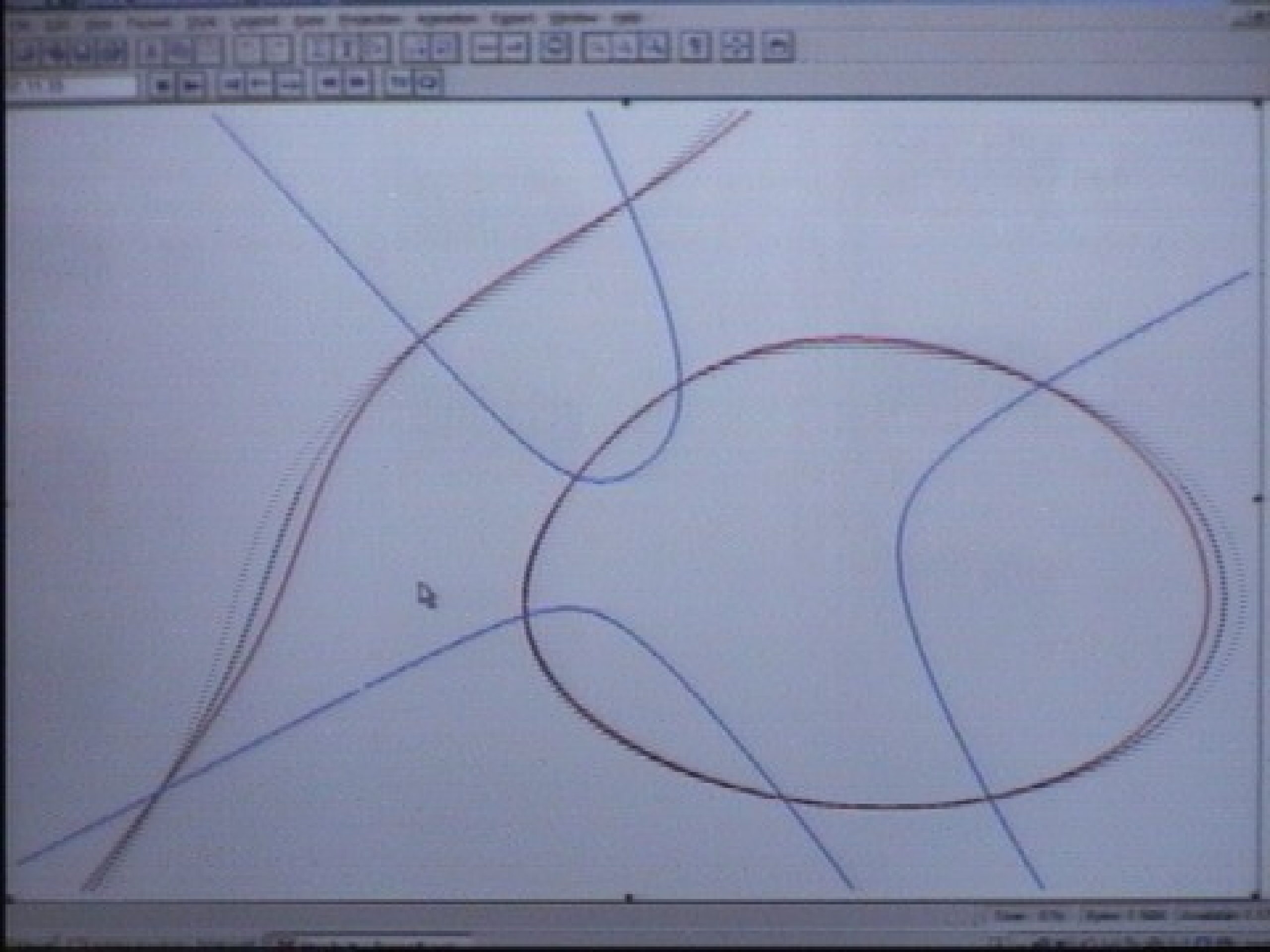

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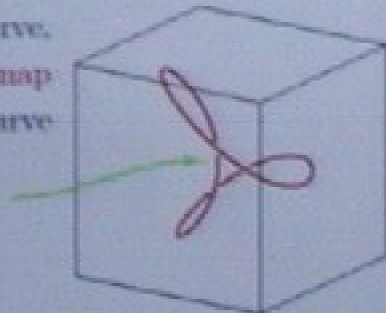
First question:

What exactly do we mean by "a curve C in X" ?

Two points of view on curves

A curve in X can be viewed as either ...

a parameterized curve,
i.e., the image of a map
from an abstract curve



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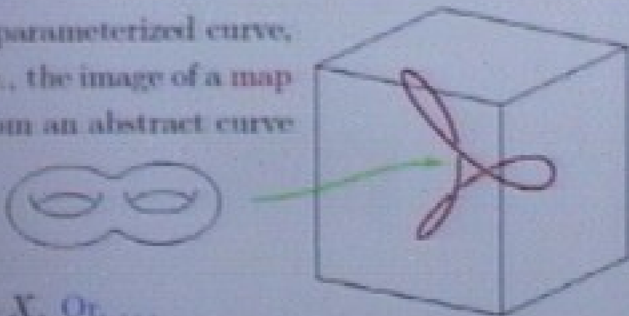

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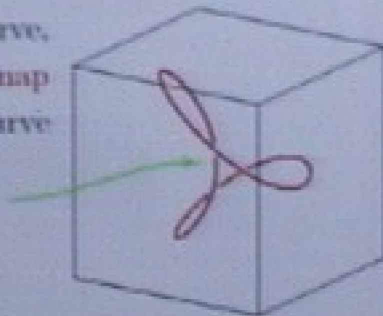

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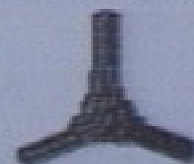

`plot(...)` vs. `implicitplot(...)`

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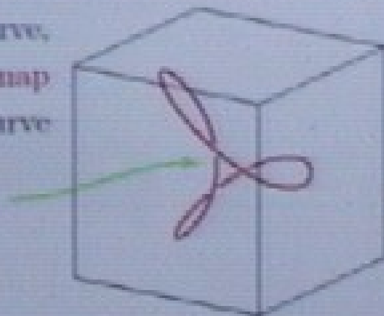

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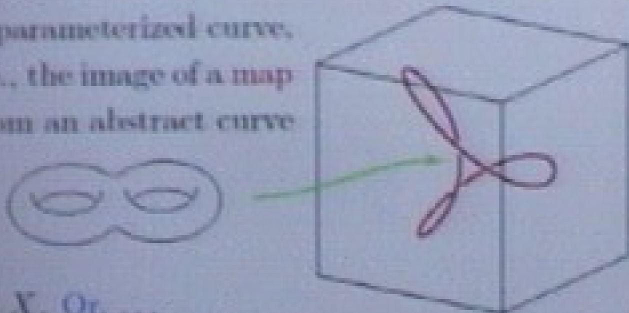

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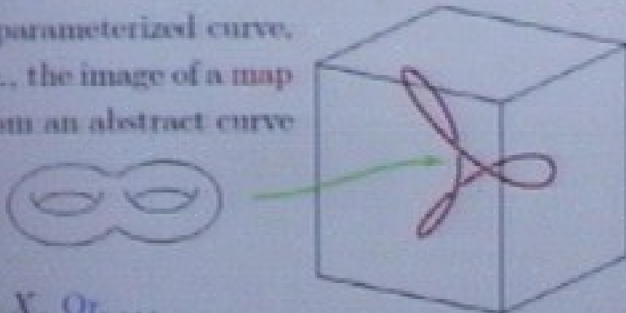

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Gromov-Witten	vs.	Donaldson-Thomas

While these two points of view are complementary for smooth curves, they differ significantly in what kind of degeneration they allow.

Prelude:

12 rational cubics meet 8 general points of the plane

curve of degree 3

and genus 0

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Degenerate objects appear in the natural compactification of the respective moduli spaces.

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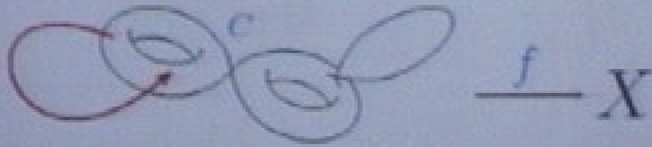
Degenerate objects appear in the natural compactification of the respective moduli spaces.

As a result, enumerative predictions of the two theories are different.

Points of the GW moduli space are

Stable maps

introduced by Kontsevich. The domain C of a stable map f is at worst nodal curve of arithmetic genus g.



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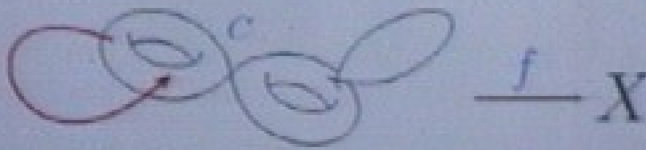

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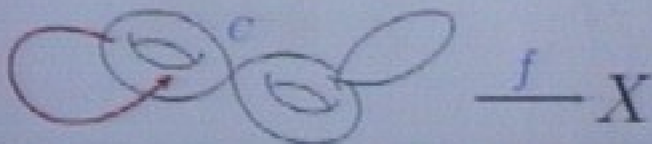

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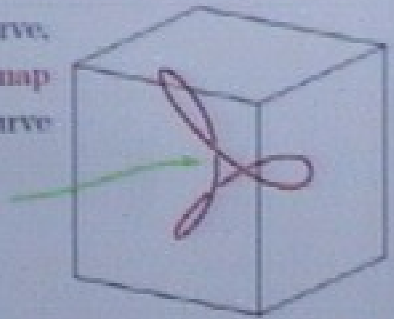

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In DT theory, a curve C are described in local charts by

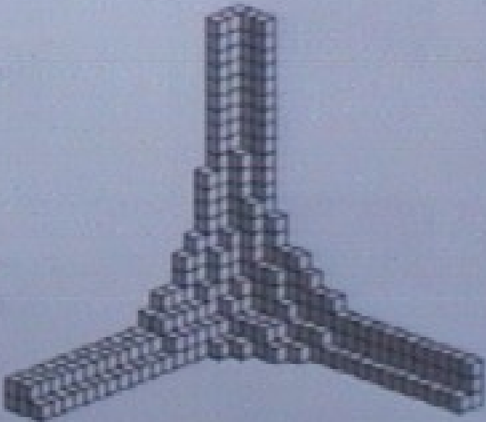
Ideals

of all polynomials F vanishing on C.

Simple and important examples of ideals

$$I \in \mathbf{C}[x, y, z]$$

are the ones spanned by monomials. These correspond to 3D partitions, because ...



Monomials ideals in 2D

1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
y	xy	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y	x^8y
y^2	xy^2	x^2y^2	x^3y^2	x^4y^2	x^5y^2	x^6y^2	x^7y^2	x^8y^2
y^3	xy^3	x^2y^3	x^3y^3	x^4y^3	x^5y^3	x^6y^3	x^7y^3	x^8y^3
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Legend:

- not in I
- in I
- generator

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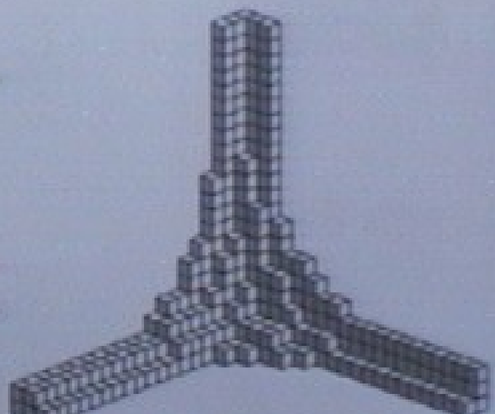

Plane cubics depend on 9 parameters. Those meeting 8 points form a 1-parameter family

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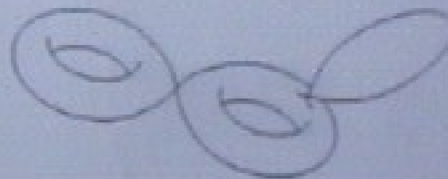

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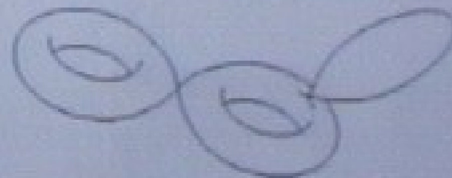

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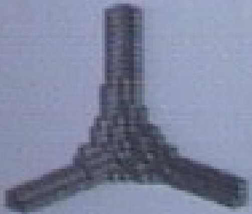

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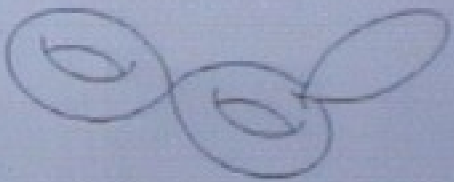

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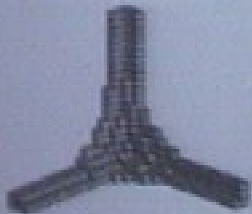

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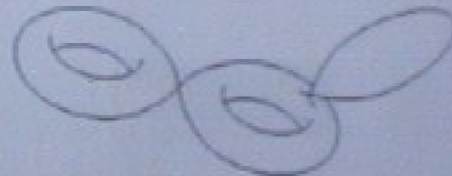

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The natural habitats of GW and DT theories are different.

And, indeed, the GW and DT enumerative predictions are different.

Fix the same degree and collection of incidence conditions γ on both GW and DT sides.

Define

$$\mathsf{GW}(u) = \sum_g u^{2g-2} \boxed{\text{GW count of genus } g \text{ curves meeting } \gamma}$$

The contribution of degree 0 maps is removed here.

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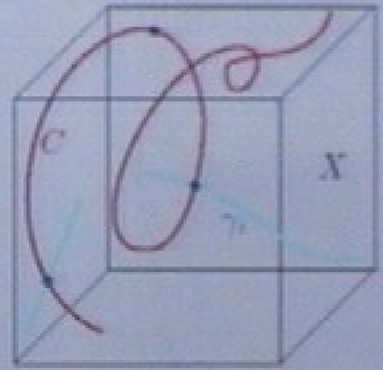
given X, e.g. projective space $\mathbb{P}^3$, how many curves $C \subset X$ of given degree and genus are incident to given cycles γ_i ?



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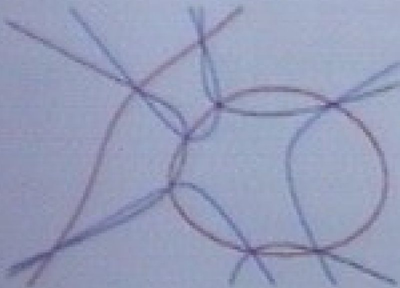

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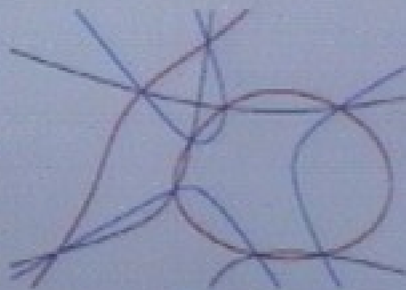
For general $t \in \mathbb{C}$ we have a curve of genus 1.

For 12 special values of t a node appears and the genus of the curve drops to 0.



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Conjecture [MNOP]

The change of variables

$$\boxed{q = -e^{iu}}$$

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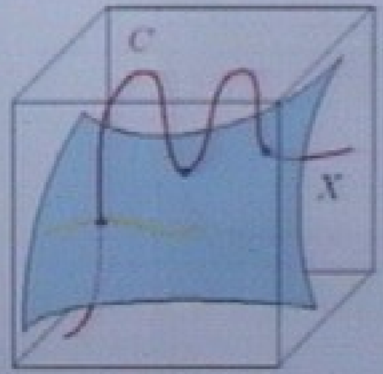
Toric means that the torus $(\mathbb{C}^*)^3$ acts on X with an open orbit. The complex projective space $\mathbb{P}^3$ is an example.

In fact, more can in be proved in the framework of

Relative theories

in which we *fix* a smooth surface $S \subset X$ and keep track of the *tangency* of C to S, i.e. the location and the multiplicity of $C \cap S$.



A relative GW/DT correspondence is conjectured.

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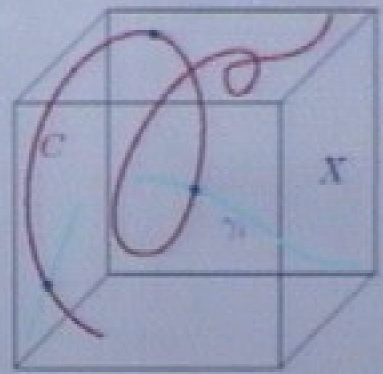
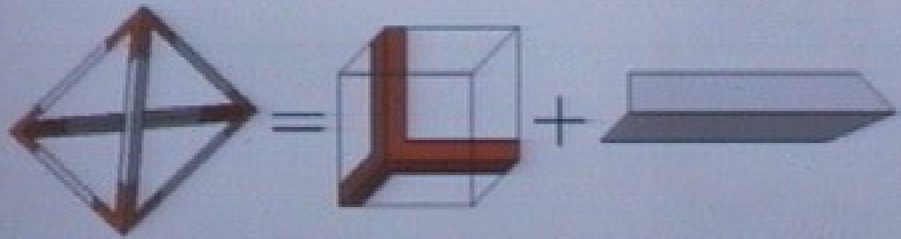


The geometry of a toric variety X is captured by its

Toric polytope

For example, for the projective space $\mathbb{P}^3$ this is a tetrahedron.

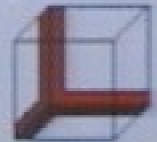
Localization and degeneration techniques (— *lectures by Vergne and Eliashberg*) allow one to treat X as if it was glued out of vertices and edges,



which represent particular equivariant relative curve counts.

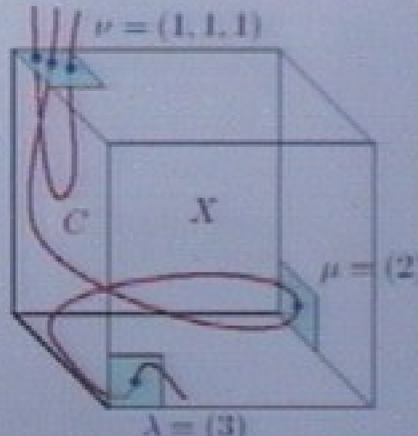
In particular,

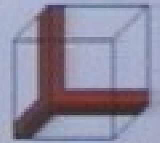
The Vertex



is shorthand for the equivariant count of curves C in an open neighborhood U of the coordinate axes in $X = (\mathbb{P}^1)^3$, relative the 3 infinities.

Tangency recorded by 3 partitions, which are summed over in the gluing formula.



The Vertex 

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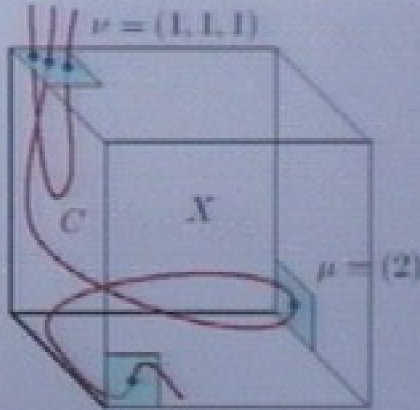
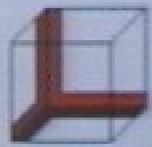


The Vertex 

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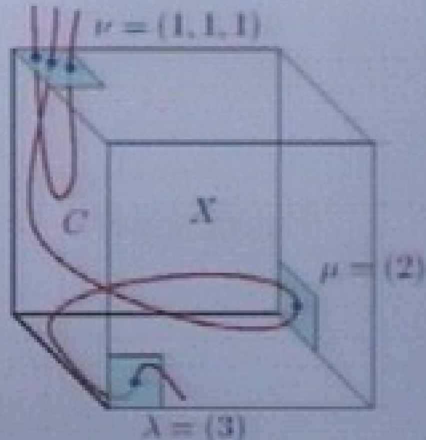

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This gives all-genera, fixed-degree counts in finite time.

< lots of formulas >

As a special case, this proves the so-called topological vertex formula, first conjectured by M. Aganagic, A. Klemm, M. Mariño and C. Vafa, and further studied by J. Li, K. Liu, M. Liu, J. Zhou, among others.

That special case leads to beautiful combinatorics.

The general case, in which the equivariant dependence remains explicit, seems much more involved.

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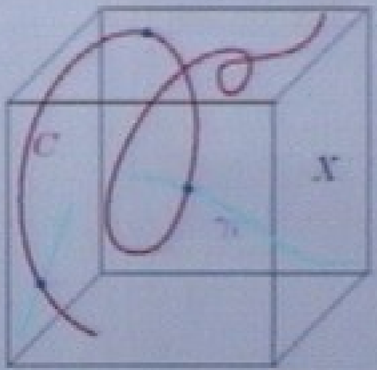

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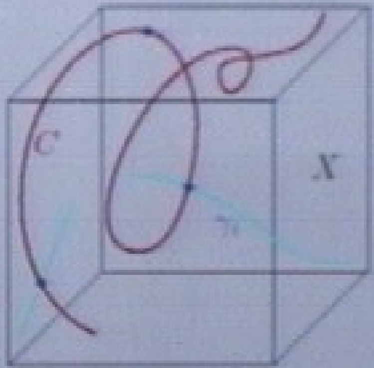

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To be honest, we care a lot more about the structure of the totality of these curve counts than about any individual number as such.

This structure appears very rich.