DEFORMATION AND RIGIDITY FOR GROUP ACTIONS AND VON NEUMANN ALGEBRAS

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- **Connes '80**: $\Gamma$ ICC with prop. T of Kazhdan then $\text{Out}(\mathcal{L}(\Gamma))$, $\mathcal{F}(\mathcal{L}(\Gamma))$ are countable!

- **Connes' Rigidity Conjecture (CRC) '80**: 
  $\Gamma, \Lambda$ ICC, prop. T, $\mathcal{L}(\Gamma) \cong \mathcal{L}(\Lambda) \Rightarrow \Gamma \cong \Lambda$? 
  Strong Version: $L^\infty X \rtimes \Gamma \cong L^\infty Y \rtimes \Lambda \Rightarrow \Gamma \cong \Lambda$ (virtually)?

**Partial Answers:**

(a) $\mathcal{L}(\Gamma) \ncong \mathcal{L}(\mathbb{F}_n)$ (Connes-Jones '84): 
Proof uses Haagerup's deformation of $\mathbb{F}_n$;

(b) $\Gamma_n \subset \text{Sp}(n, 1)$ lattice, then $\mathcal{L}(\Gamma_n) \cong \mathcal{L}(\Gamma_m)$ 
$\Rightarrow n \leq m$ (Cowling-Haagerup '85);

(c) Strong CRC true modulo countable sets, 
i.e. $\Gamma \cong L\Gamma$ countable to 1 (Popa '06). 
Proof by "separability arguments" + results of Gromov and Shalom.
Meanwhile in OE Ergodic Theory

R. Zimmer '80: $SL(n, \mathbb{Z}) \bowtie X$, $SL(m, \mathbb{Z}) \bowtie Y$ free ergodic OE $\Rightarrow n = m$.
Proof uses Zimmer's cocycle superrigidity

D. Gaboriau '96-'01: $\mathbb{F}_n \bowtie X$, $\mathbb{F}_m \bowtie Y$ free ergodic OE $\Rightarrow n = m$. Also $\mathcal{F}(\mathcal{R}_{\mathbb{F}_n}) = \{1\}$, $2 \leq n < \infty$.
Proof uses Gaboriau's $\ell^2$-Betti numbers for equiv. relations $\beta_n(\mathcal{R}) \in [0, \infty]$, for which he shows:
$\beta_n(\mathcal{R}_\Gamma) = \beta_n(\Gamma)$ (Atiyah, Cheeger-Gromov)
$\beta_n(\mathcal{R}^t) = \beta_n(\mathcal{R})/t$.

A. Furman '99: Free ergodic $\Gamma \bowtie X$ with $\Gamma$ higher rank lattice are OE Superrigid. any OE between $\Gamma \bowtie X$ and an arbitrary free ergodic $\Lambda \bowtie Y$ comes from a conjugacy.
Proof uses Zimmer and Ratner results.

N. Monod & Y. Shalom '02: OE Superrigidity for products of $\geq 2$ word-hyperbolic groups.
Proof uses bounded cohom. (Burger-Monod).
vN and OE rigidity from coexistence of deformation & prop. T

**Thm 1 (P '01).** $\Gamma, \Lambda \subset SL(2, \mathbb{Z})$ non-amenable, $\Gamma, \Lambda \curvearrowright T^2 = \mathbb{R}^2$. Then:
\[ \forall L^\infty T^2 \rtimes \Gamma \simeq L^\infty T^2 \rtimes \Lambda \text{ comes from OE}. \]

Proof uses deformation/rigidity and intertwining subalgebras techniques.

Consequences of Thm 1 + Gaboriau's results:

- $\mathcal{F}(L^\infty T^2 \rtimes \Gamma) = \{1\}, \forall \Gamma \subset SL(2, \mathbb{Z})$ fin. index
- $L^\infty T^2 \rtimes F_n, n = 2, 3, \ldots$ non-isomorphic.
Terminology: \( \Gamma \) \textbf{w-rigid} if \( \exists H \subseteq \Gamma \) normal with relative prop. \( \mathcal{T} \) of Kazhdan-Margulis.
- \( \Gamma = \Gamma_0 \rtimes \mathbb{Z}^2 \) for \( \Gamma_0 \subseteq SL(2, \mathbb{Z}) \) non-amenable (Burger);
- \( \Gamma = H \times H' \) with \( H \) infinite Kazhdan.

\textbf{Thm 2 (P '01-'04).} \( \Gamma \) w-rigid ICC, \( \Gamma \bowtie X \) free ergodic; \( X \) arbitrary ICC, \( \bowtie \gamma N \) Bernoulli. If \( \rho : L^\infty X \times \Gamma \rightarrow (L^\infty Y \times \Lambda)^\Gamma \) then \( \epsilon = 1 \)
and \( \rho \) comes from a conjugacy.

In particular:
- \( \Gamma \) w-rigid ICC, \( \Gamma \bowtie X \) Bernoulli, then \( M = L^\infty X \rtimes \Gamma \) has \( \mathcal{F}(M) = 1 \), \( \text{Out}(M) \) calculable.
- \( \bowtie \) CRC for wreath products: \( \Gamma_0 \bowtie \) w-rigid ICC, \( H \) discrete abelian, \( G_0 = H \bowtie \Gamma_0 \) (wreath prod.), \( \epsilon = 0, 1 \). Then: \( LG_0 \bowtie LG_1 \) iff \( G_0 \bowtie G_1 \).
Thm 3 (Ioana-Peterson-P '05). 
\( \forall K \) compact abelian, \( n, m \geq 3 \), \( \exists \) action 
\( \Gamma = SL(n, \mathbb{Z}) \ast (SL(m, \mathbb{Z}) \times K) \curvearrowright \mathbb{R} \) with 
\( \mathcal{F}(\mathbb{R} \times \Gamma) = \{1\} \) and \( \text{Out}(\mathbb{R} \times \Gamma) = K \).

Thm 4 (P-Vaes '06). \( \Gamma = SL(4, \mathbb{Z}) \times \mathbb{Z}^A \), 
\( \Gamma_0 = \{ \pm A^n \mid n \in \mathbb{Z} \} \), for certain \( A \in SL(4, \mathbb{Z}) \)
\( \Gamma \curvearrowright (X, \mu) = ([0, 1], \mu_0)_{\Gamma/\Gamma_0}, \mu_0(0) \neq \mu_0(1) \).
Then \( M = L^\infty X \times \Gamma \) has no "outer symmetries": \( \mathcal{F}(M) = 1 \), \( \text{Out}(M) = 1 \), \( M \not\cong M^{op} \).
Thm 5 (P '05): OE Superrigidity
Assume $\Gamma$ w-rigid, $\Gamma \curvearrowright X$ Bernoulli. Then $\Gamma \curvearrowright X$ is OE superrigid:
\[ \forall \text{OE between } \Gamma \curvearrowright X \text{ and arbitrary free ergodic } \Lambda \curvearrowright Y \text{ comes from a conjugacy} \]

Proof follows from cocycle superrigidity below, applied to cocycle assoc. to the OE (Zimmer).

Terminology: Closed subgroups $\mathcal{U} \subset U(N)$ with $N$ II$_1$ factor called finite type.
E.g.: $\forall$ countable discrete, $\forall$ separable compact groups.

Thm 6 (P '05): Cocycle Superrigidity
Assume: $\Gamma$ w-rigid, $\Gamma \curvearrowright X$ Bernoulli. Then:
$\forall \, \mathcal{U}$ finite type, $\forall \mathcal{U}$-valued cocycle for $\Gamma \curvearrowright X$ can be "untwisted" to a group morphism $\Gamma \to \mathcal{U}$. 
Applying the same methods to OE Ergodic Theory, gives:
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**Thm 6 (P ‘05): OE Superrigidity**

Assume $\Gamma$ w-rigid, $\Gamma \bowtie X$ Bernoulli. Then $\Gamma \bowtie X$ is OE superrigid:

$\forall$ OE between $\Gamma \bowtie X$ and arbitrary free ergodic $\Lambda \bowtie Y$ comes from a conjugacy.

Even more so: $\forall \Lambda \bowtie Y$ free ergodic, if $\Delta : X \cong Y$ satisfies $\Delta(R_\Gamma) \subset R_\Lambda$ (takes orbits of $\Gamma$ into orbits of $\Lambda$) then $\exists \Lambda_0 \subset \Lambda$ and $\alpha \subset R_\Lambda$ such that $\Delta_0 = \alpha \circ \Delta$ satisfies $\Delta_0 \cap \Delta_0^{-1} = \Lambda_0$. 
Applying the same methods to OE Ergodic Theory, gives:

**Thm 6 (P '05): OE Superrigidity**

Assume $\Gamma$ w-rigid, $\Gamma \act X$ Bernoulli. Then $\Gamma \act X$ is OE superrigid:

$\forall$ OE between $\Gamma \act X$ and arbitrary free ergodic $\Lambda \act Y$ comes from a conjugacy.

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Terminology: Closed subgroups $V \subset H(N)$ with $N$ II$_1$ factor called finite type. E.g.: $V$ countable discrete, $V$ separable compact groups.

**Thm 7 (P '05): Cocycle Superrigidity**
Assume: $\Gamma$ w-rigid, $\Gamma \curvearrowright X$ Bernoulli. Then: $\forall V$ finite type, $\forall V$-valued cocycle for $\Gamma \curvearrowright X$ can be "untwisted" to a group morphism $\Gamma \to V$.

**Thm 7' (P '06):** Same holds true if $\Gamma$ has two infinite commuting subgroups $H, H' \subset \Gamma$, one of which non-amenable, such that $H H'$ is "weakly normal" in $\Gamma$.

*Proof* uses similar deformation/rigidity strategy, and holds true for all malleable actions (not only Bernoulli actions).

[Subgroup $G \subset \Gamma$ is weakly normal if $\Gamma$ generated by $\{g \in \Gamma \mid |gGg^{-1} \cap G| = \infty\}$.]
Isomorphism of probability measure spaces

\[ \Delta : (X, \mu) \simeq (Y, \nu) \]

viewed also as isomorphism of algebras.

\[ \Delta : (L^\infty X, \int \cdot d\mu) \simeq (L^\infty Y, \int \cdot d\nu) \]

via the relation \( \Delta a(t) = a(\Delta^{-1} t) \), \( \forall a \in L^\infty X = L^\infty(X), \ t \in X \). Extends to \( L^2 X \simeq L^2 Y \).

\( \text{Aut}(X, \mu) \) "same as" \( \text{Aut}(L^\infty X, \int \cdot d\mu) \)
\( \Gamma \) group: a measure preserving \( \Gamma \)-action is a morphism \( \Gamma \to \text{Aut}(X, \mu) \), or \( \Gamma \to \text{Aut}(L^\infty X, \mu) \).

Notation: \( \Gamma \acts X, \Gamma \acts L^\infty X \)

\( \Gamma \acts X \) free if: \( \mu(\{t \in X \mid g \cdot t = t\}) = 0, \forall g \neq e \)

ergodic if: \( \exists \in L^\infty X, g \cdot \exists = \exists \Rightarrow g \in \mathcal{C}_1 \).

Example 1: \( \mathcal{T} \in \text{Aut}(X, \mu) \) (e.g. irrational rotation on \( X = \mathcal{T} \); two-sided Bernoulli shift on \( X = \{0,1\}^\mathbb{Z} \), etc), dynamical system \( (\mathcal{T}^\mathcal{m}, \mathcal{n} \in \mathbb{Z}) \) same as \( \mathcal{Z} \acts X \).

Example 2 (Bernoulli actions): \( \Gamma \) countable grp. \( (X_0, \mu_0) \) prob space, \( \Gamma \acts (X, \mu) = (X_0, \mu_0)\Gamma \)
by \( g(t_h)_h = (t_{g^{-1}h})_h \). Similarly \( \Gamma \acts (X_0, \mu_0)\Gamma / \Gamma_0 \), for \( \Gamma_0 \subset \Gamma \) subgroup.

Example 3 ("Geometric" actions):
(a) \( \Gamma, \Lambda \) lattices in Lie group \( G \), \( \Gamma \acts G / \Lambda \);
(b) \( \Gamma \subset GL(n, \mathbb{Z}) \), \( \Gamma \acts \mathbb{Z}^n = \mathbb{Z}^n \);
(c) \( \Gamma \leftarrow G \) dense in compact group \( G \), \( \Gamma \acts G \).
The group measure space construction  
(F. Murray & J. von Neumann ’36)  

\[ \Gamma \curvearrowright (X, \mu) \]

von Neumann Algebras  
\[ L^\infty X \rtimes \Gamma, \mathcal{L}(\Gamma) \]

- \( \mathcal{H} = \oplus_g L^2X_{ug} \) Hilbert space
- \( \mathcal{H} \ni \sum_h \xi_h u_h \) "series"
- \( L^2X \triangleq L^2(X) \ni \xi_h \) "coefficients"
- \( u_h, h \in \Gamma \), "variables"
- Multiplication: \( (x_g u_g)(\xi_h u_h) = x_g \phi(\xi_h) u_{gh} \)

Then:
\[ L^\infty X \rtimes \Gamma \overset{\text{def}}{=} \{ x \in \mathcal{H} \mid x \xi \in \mathcal{H}, \forall \xi \in \mathcal{H} \} \]
\[ \mathcal{L}(\Gamma) \overset{\text{def}}{=} \mathbb{C} \rtimes \Gamma \]

as algebras of left multiplication operators on \( \mathcal{H} \). They are von Neumann algebras, i.e. closed in topology given by seminorms \( \| \langle \xi, \eta \rangle \| \) on \( \mathcal{B}(\mathcal{H}) \).
- $L^\infty X$ as subalgebra of $M = L^\infty X \times \Gamma$ by
  $a \mapsto a\omega_e = a1$
- $\int d\mu$ extends to positive linear functional $\tau$ on $M$ by
  $\tau(\Sigma g_0 \omega_g u_g) = \int \omega_g d\mu$. Satisfies $\tau(\omega y) = \tau(y\omega)$,
  $\forall \omega, y$, i.e. $\tau$ trace on $M$.

- $\Gamma \actson (X, \mu)$ free ergodic, $|\Gamma| = n < \infty$, then:
  $X \simeq \{1, ..., n\}$, $\mu$ the counting measure
  $L^\infty X \times \Gamma \simeq M_{n \times n}(\mathbb{C})$, $\tau = Tr(\cdot)/n$
  $L^\infty X$ = diagonal operators.

- $\Gamma \actson (X, \mu)$ free ergodic, $|\Gamma| = \infty$, then $M$
  II$_1$ factor, i.e. $Z(M) = \mathbb{C}$, $M$ has unique trace,
  $\tau(\mathcal{P}(M)) = [0, 1]$ ("continuous dimension")

- $\mathcal{L}(\Gamma)$ is II$_1$ factor iff $\Gamma$ infinite conjugacy class (ICC).
  E.g.: $\Gamma = S_{\infty}, \mathbb{F}_n, PSL(n, \mathbb{Z})$, $n \geq 2$. 
Continuous dimension allows $t$-amplification of II$_1$ factor $M$, $\forall t > 0$, by $M^t = pM_{\infty}(M)p$, $n \geq t$, $p \in \mathcal{P}(M_{\infty}(M))$, $\tau(p) = t/n$.

Notice: $(M^t)^s = M^{ts}$.

Fundamental group of $M$:
\[
\mathcal{F}(M) \overset{\text{def}}{=} \{ t > 0 \mid M^t \simeq M \}.
\]

Murray & von Neumann '43:
All II$_1$ factors $L^\infty X \rtimes \Gamma$ with $\Gamma$ increasing union of finite groups is approximately finite dim. (AFD) and all AFD II$_1$ factors are isomorphic.

The unique AFD factor denoted $R$.

Consequence: $\mathcal{F}(R) = \mathbb{R}_+^*$. 
How does the (stable) isomorphism class of $M = L^\infty X \rtimes \Gamma$ depend on $\Gamma \varsubsetneq X$? Specifically: Describe $L^\infty X \rtimes \Gamma \simeq (L^\infty Y \rtimes \Lambda)^f$ in terms of "isomorphisms" of $\Gamma \varsubsetneq X$, $\Lambda \varsubsetneq Y$. In particular:

- Calculate $\text{Out}(M) \overset{\text{def}}{=} \text{Aut/Int}(M)$ and $\mathcal{F}(M)$, for $M = L^\infty X \rtimes \Gamma$.

"Classic Questions"
(Murray-von Neumann '43, Kadison '67):
$\exists \Gamma \varsubsetneq X$, $\mathcal{F}(L^\infty X \rtimes \Gamma) = 1$? $\mathcal{L}(\mathcal{F}_n) \simeq \mathcal{L}(\mathcal{F}_m) \Rightarrow n = m$? $\mathcal{F}(\mathcal{L}(\mathcal{F}_n)) = ?$ More generally:
$L^\infty X \rtimes \mathcal{F}_n \simeq L^\infty X \rtimes \mathcal{F}_m \Rightarrow n = m$?
$\mathcal{F}(L^\infty X \rtimes \mathcal{F}_n) =$?

"Isomorphism" of $\Gamma \varsubsetneq X$, $\Lambda \varsubsetneq Y$ means conjugacy, i.e. $\Delta : (X, \mu) \simeq (Y, \nu)$ and $\delta : \Gamma \simeq \Lambda$ with $\Delta(\delta t) = \delta(g)\Delta(t)$, $\forall g \in \Gamma, t \in X$.

Note: Conjugacy implements isomorphism $L^\infty X \rtimes \Gamma \simeq L^\infty Y \rtimes \Lambda$ by $\sum a_g u_g \mapsto \sum \Delta(a_g) v_{q(g)}$. 

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\item How does the (stable) isomorphism class of $M = L^\infty X \rtimes \Gamma$ depend on $\Gamma \varsubsetneq X$? Specifically: Describe $L^\infty X \rtimes \Gamma \simeq (L^\infty Y \rtimes \Lambda)^f$ in terms of "isomorphisms" of $\Gamma \varsubsetneq X$, $\Lambda \varsubsetneq Y$. In particular:
\begin{itemize}
\item Calculate $\text{Out}(M) \overset{\text{def}}{=} \text{Aut/Int}(M)$ and $\mathcal{F}(M)$, for $M = L^\infty X \rtimes \Gamma$.
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$\exists \Gamma \varsubsetneq X$, $\mathcal{F}(L^\infty X \rtimes \Gamma) = 1$? $\mathcal{L}(\mathcal{F}_n) \simeq \mathcal{L}(\mathcal{F}_m) \Rightarrow n = m$? $\mathcal{F}(\mathcal{L}(\mathcal{F}_n)) = ?$ More generally:
$L^\infty X \rtimes \mathcal{F}_n \simeq L^\infty X \rtimes \mathcal{F}_m \Rightarrow n = m$?
$\mathcal{F}(L^\infty X \rtimes \mathcal{F}_n) =$?

"Isomorphism" of $\Gamma \varsubsetneq X$, $\Lambda \varsubsetneq Y$ means conjugacy, i.e. $\Delta : (X, \mu) \simeq (Y, \nu)$ and $\delta : \Gamma \simeq \Lambda$ with $\Delta(\delta t) = \delta(g)\Delta(t)$, $\forall g \in \Gamma, t \in X$.

Note: Conjugacy implements isomorphism $L^\infty X \rtimes \Gamma \simeq L^\infty Y \rtimes \Lambda$ by $\sum a_g u_g \mapsto \sum \Delta(a_g) v_{q(g)}$. 

\end{itemize}
Singer '55: $L^\infty X \times \Gamma$ can only "remember" the equivalence relation given by its orbits of $\Gamma \curvearrowright X$: $R_\Gamma \overset{\text{def}}{=} \{(t, g\cdot t) \mid t \in X, g \in \Gamma\}$.

Equivalently, Feldman-Moore '77: An iso $\Delta : (X, \mu) \simeq (Y, \nu)$ extends to $L^\infty X \times \Gamma \simeq L^\infty Y \times \Lambda$ iff $\Delta$ is an orbit equivalence (OE), i.e. $\Delta(R_\Gamma) = R_\Lambda$, or $\Delta(\Gamma\cdot t) = \Lambda(\Delta(t))$, $\forall t$.

Thus: Conjugacy $\Rightarrow$ OE $\Rightarrow$ iso of vn algebras.

Orbit Equivalence Ergodic Theory:
(H. Dye '59, FM '77)
Study of $\Gamma \curvearrowright X$ up to OE, or:
How does OE class of $\Gamma \curvearrowright X$ depend on its conjugacy class, in particular on group $\Gamma$?
• Connes '76: All amenable II_1 factors are isomorphic to \( R \). II_1 factors \( L^\infty X \rtimes \Gamma \), \( L(\Gamma) \) amenable (thus \( \cong R \)) iff \( \Gamma \) amenable.

• Dye '59, Ornstein-Weiss, Connes-Feldman-Weiss '81: All ergodic m.p. actions of countable groups on non-atomic prob. spaces are OE.

• \( \forall \Gamma \) non-amenable has \( \geq 2 \) non OE free ergodic actions (K. Schmidt, Connes-Weiss '81; for non-\( T \); Hjorth '02: \( \forall T \)-group \( \exists \infty \) non OE; Gaboriau-Popa '03 same for \( \mathbb{F}_n \); Golodets '83, Monod-Shalom, Popa, Ioana '02-'04 same for many non-amenable groups).

• Connes-Jones '82: \( \exists \) group \( \Gamma \) and free ergodic \( \Gamma \rtimes X \), \( \Gamma \rtimes Y \) non OE but give same algebra \( L^\infty X \rtimes \Gamma \cong L^\infty Y \rtimes \Gamma \).