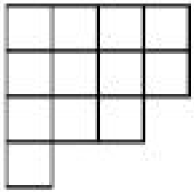
OK

**partition** $\lambda \vdash n$: $\lambda = (\lambda_1, \lambda_2, \ldots)$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

$$\sum \lambda_i = n$$

**(Young) diagram** of $\lambda = (4, 4, 3, 1)$:



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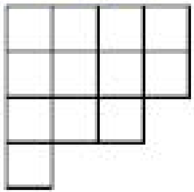
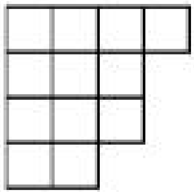


Young diagram of the **conjugate** partition $\lambda' = (4, 3, 3, 2)$:



**standard Young tableau** (SYT) of shape $\lambda \vdash n$, e.g., $\lambda = (4, 4, 3, 1)$:

$<$

$\wedge$

| | | | |
|---|---|---|---|
| 1 | 2 | 7 | 10 |
| 3 | 5 | 8 | 12 |
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| 9 | | | |

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$f^\lambda = \#$ of SYT of shape $\lambda$

E.g., $f^{(3,2)} = 5$:

| | | | | |
|---|---|---|---|---|
| 123 | 124 | 125 | 134 | 135 |
| 45 | 35 | 34 | 25 | 24 |

$\exists$ simple formula for $f^\lambda$ (Frame-Robinson-Thrall **hook-length formula**)

**Note.** $f^\lambda = \dim(\text{irrep. of } \mathfrak{S}_n)$, where $\mathfrak{S}_n$ is the **symmetric group** of all permutations of $1, 2 \ldots, n$.

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**RSK algorithm:** a bijection

$$w \xrightarrow{\text{rsk}} (P, Q),$$

where $w \in \mathfrak{S}_n$ and $P, Q$ are SYT of the same shape $\lambda \vdash n$.

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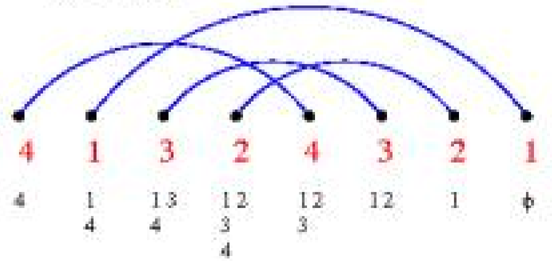
**R** = Gilbert de Beauregard Robinson
**S** = Craige Schensted (= Ea Ea)
**K** = Donald Ervin Knuth

$w = 4132$:



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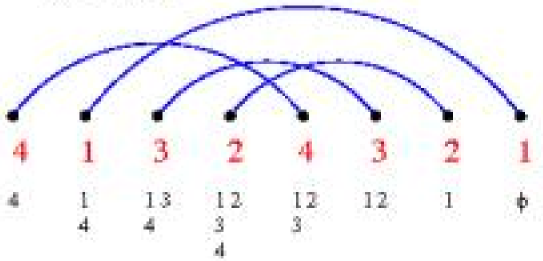


$$(P, Q) = \begin{pmatrix} \begin{matrix} 1\,2 \\ 3 \\ 4 \end{matrix} & , & \begin{matrix} 1\,3 \\ 2 \\ 4 \end{matrix} \end{pmatrix}$$

**Schensted's theorem:** *Let* $w \xrightarrow{\text{rsk}} (P, Q)$, *where* $sh(P) = \text{sh}(Q) = \lambda$. *Then*

$$\text{is}(w) = \text{longest row length} = \lambda_1$$
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Increasing and decreasing subsequences [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

partition [illegible]

[illegible]

(Young) diagram of [illegible]

Young diagram of the conjugate partition [illegible]

standard Young tableau (SYT) of shape [illegible]

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$$4132 \xrightarrow{\text{rsk}} \begin{pmatrix} 1\,2 & & 1\,3 \\ 3 & , & 2 \\ 4 & & 4 \end{pmatrix}$$

$$\mathrm{is}(w) = 2, \quad \mathrm{ds}(w) = 3$$

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**Corollary** (Erdős-Szekeres, Seidenberg). *Let* $w \in \mathfrak{S}_{pq+1}$*. Then either* $\text{is}(w) > p$ *or* $\text{ds}(w) > q$*.*

**Proof.** Let $\lambda = \text{sh}(w)$. If $\text{is}(w) \leq p$ and $\text{ds}(w) \leq q$ then $\lambda_1 \leq p$ and $\lambda'_1 \leq q$, so $\sum \lambda_i \leq pq$. $\square$

**Corollary.** *Say $p \le q$. Then*

$$\#\{w \in \mathfrak{S}_{pq} : \operatorname{is}(w) = p,\ \operatorname{ds}(w) = q\} = \left(f^{(p^q)}\right)^2$$

By hook-length formula, this is

$$\left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1}\right)^2.$$

**Romik**: let

$$w \in \mathfrak{S}_{p^2}, \quad \operatorname{is}(w) = \operatorname{ds}(w) = p.$$

Let $P_w$ be the permutation matrix of $w$ with corners $(\pm 1, \pm 1)$. Then (informally) as $p \to \infty$ almost *surely* the 1's in $P_w$ will become dense in the region bounded by the curve

$$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3,$$

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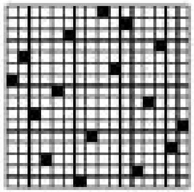
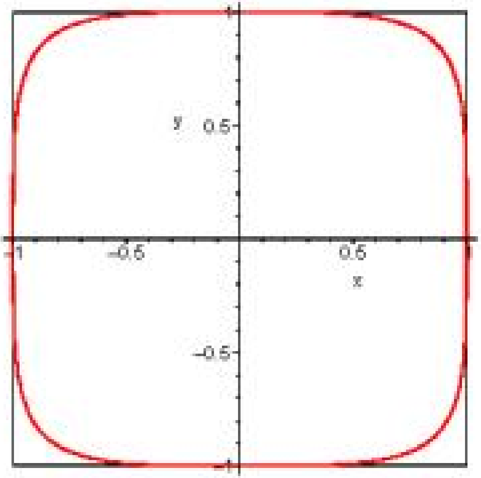


$w = 9, 11, 6, 14, 2, 10, 1, 5, 13, 3, 16, 8, 15, 4, 12, 7$

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## Distribution of is($w$)

$$\begin{aligned} E(n) &= \text{expectation of is}(w),\ w \in \mathfrak{S}_n \\ &= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2 \end{aligned}$$

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**Ulam**: what is distribution of is($w$)? rate of growth of $E(n)$?

**Hammersley** (1972):

$$\exists\ c = \lim_{n\to\infty} \frac{E(n)}{\sqrt{n}},$$

and

$$\frac{\pi}{2} \le c \le e.$$

Conjectured $c = 2$.

**Logan-Shepp, Vershik-Kerov** (1977): $c = 2$

**Idea of proof.**

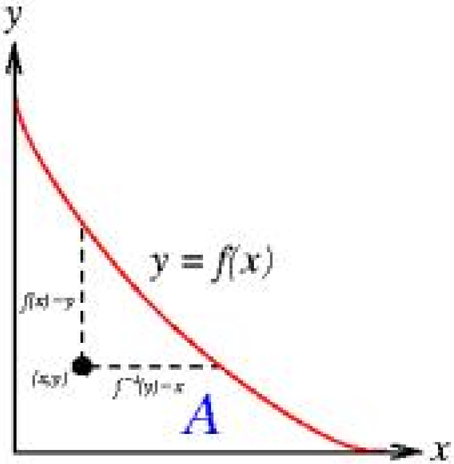
$$E(n) = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2$$
$$\approx \frac{1}{n!} \max_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2.$$

Find "limiting shape" of $\lambda \vdash n$ maximizing $\lambda$ as $n \to \infty$ using hook-length formula.

## Increasing and decreasing subsequences

$$3\,1\,8\,4\,9\,6\,7\,2\,5 \quad \text{(i.s.)}$$
$$3\,1\,8\,4\,9\,6\,7\,2\,5 \quad \text{(d.s.)}$$

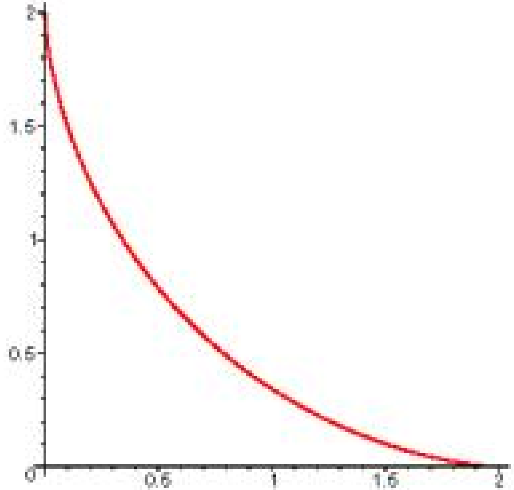
$$\mathbf{is}(w) = |\text{longest i.s.}| = 4$$
$$\mathbf{ds}(w) = |\text{longest d.s.}| = 3$$

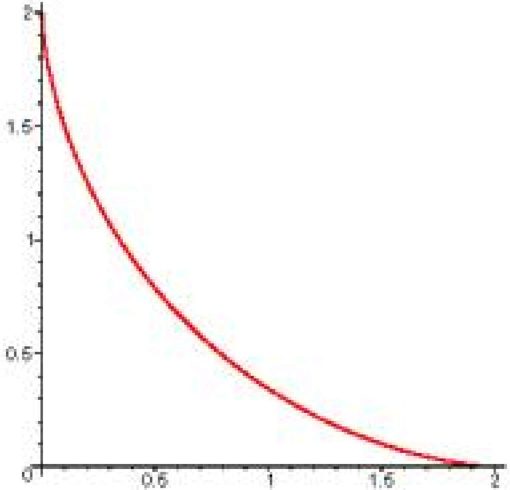



$$\min \iint_A \log(f(x) + f^{-1}(y) - x - y)\,dx\,dy,$$

subject to

$$\iint_A dx\,dy = 1.$$


2
1.5
1
0.5
0
0.5
1
1.5
2




$$x = y + 2\cos\theta$$

$$y = \frac{2}{\pi}(\sin\theta - \theta\cos\theta)$$

$$0 \le \theta \le \pi$$

$$u_k(n) := \#\{w \in \mathfrak{S}_n : \mathrm{is}_n(w) \le k\}.$$

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For $\geq$130 combinatorial interpretations of $C_n$, see

`www-math.mit.edu/~rstan/ec`

I. Gessel (1990):

$$\sum_{n\geq 0} u_k(n)\frac{x^{2n}}{n!^2} = \det\left[I_{|i-j|}(2x)\right]_{i,j=1}^{k},$$

where

$$I_m(2x) = \sum_{j\geq 0}\frac{x^{m+2j}}{j!\,(m+j)!},$$

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E.g.,

$$\sum_{n\geq 0} u_2(n)\frac{x^{2n}}{n!^2} = U_0(2x)^2 - U_1(2x)^2$$

$$= \sum_{n\geq 0} C_n\frac{x^{2n}}{n!^2}.$$

**Corollary**. For fixed $k$, $u_k(n)$ is **P-recursive**, e.g.,

$$
\begin{aligned}
&(n+4)(n+3)^2 u_4(n) \\
&= (20n^3+62n^2+22n-24)u_4(n-1) \\
&\quad -64n(n-1)^2 u_4(n-2)
\end{aligned}
$$

$$
\begin{aligned}
&(n+6)^2(n+4)^2 u_5(n) \\
&= (375-400n-843n^2-322n^3-35n^4)u_5(n-1) \\
&\quad +(259n^2+622n+45)(n-1)^2 u_5(n-2) \\
&\quad -225(n-1)^2(n-2)^2 u_5(n-3).
\end{aligned}
$$

Conjectures on form of recurrence due to Bergeron, Favreau, and Krob.

**Baik-Deift-Johansson:**

Define $u(x)$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

with certain initial conditions.

$(*)$ is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

**Application:** airplane boarding

**Naive model:** passengers board in order $w = a_1 a_2 \cdots a_n$ for seats $1, 2, \ldots, n$. Each passenger takes one time unit to be seated after arriving at his seat.

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## Tracy-Widom distribution:

$$\boldsymbol{F(t)}$$

$$= \exp\left( - \int_t^\infty (x-t) u(x)^2 \, dx \right)$$

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**Theorem** (Baik-Deift-Johansson) *For random (uniform) $w \in \mathfrak{S}_n$ and all $t \in \mathbb{R}$ we have*

$$\lim_{n\to\infty} \operatorname{Prob}\left(\frac{\operatorname{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \le t\right) = F(t).$$

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**Corollary.**

$$\operatorname{is}_n(w) = 2\sqrt{n} + \left(\int t\,dF(t)\right) n^{1/6} + o(n^{1/6})$$

$$= 2\sqrt{n} - (1.7711\cdots)n^{1/6} + o(n^{1/6})$$

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Where did the Tracy-Widom distribution $F(t)$ come from?

---

$$F(t)$$

$$= \exp\left(-\int_t^{\infty} (x-t)u(x)^2\,dx\right)$$

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

**Application:** airplane boarding

**Naive model:** passengers board in order $w = a_1 a_2 \cdots a_n$ for seats $1, 2, \ldots, n$. Each passenger takes one time unit to be seated after arriving at his seat.

**Gaussian Unitary Ensemble** (GUE):

Consider an $n \times n$ hermitian matrix $\boldsymbol{M} = (\boldsymbol{M_{ij}})$ with probability density

$$Z_n^{-1} e^{-\operatorname{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii} \cdot \prod_{i<j} d(\operatorname{Re}(M_{ij}))\, d(\operatorname{Im}(M_{ij})),$$

where $\boldsymbol{Z_n}$ is a normalization constant.

**Tracy-Widom** (1994): let $\alpha_1$ denote the largest eigenvalue of $M$. Then

$$\lim_{n\to\infty} \mathrm{Prob}\left(\left(\alpha_1 - \sqrt{2n}\right)\sqrt{2}n^{1/6} \le t\right) = F(t).$$

Is the connection between $\mathrm{is}(w)$ and GUE a coincidence?

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Okounkov provides a connection, via the theory of **random topologies on surfaces**. Very briefly, a surface can be described in two ways:

- Gluing polygons along their edges, connected to random matrices via quantum gravity.
- Ramified covering of a sphere, which can be formulated in terms of permutations.

Joint with:

Bill Chen 陈永川

Eva Deng 邓玉平

Rosena Du 杜若霞

Catherine Yan 颜华菲

(complete) matching:

**crossing:** 

**nesting:** 

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total number of matchings on $[2n] := \{1, 2, \ldots, 2n\}$ is

$$\mathbf{(2n-1)!!} := 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

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total number of matchings on $[2n] := \{1, 2, \ldots, 2n\}$ is

$$(2n-1)!! := 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

**Theorem.** *The number of matchings on $[2n]$ with no crossings (or with no nestings) is*

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

**Well-known:**

$$C_n = \#\{a_1 \cdots a_{2n} : a_i = \pm 1,$$
$$a_1 + \cdots + a_i \geq 0,\ \sum a_i = 0\}$$

(**ballot sequence**).

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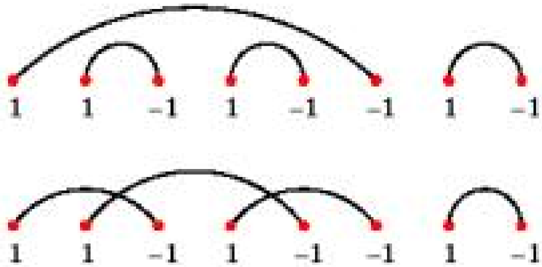

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What is the analogue of increasing and decreasing subsequences for matchings $M$?

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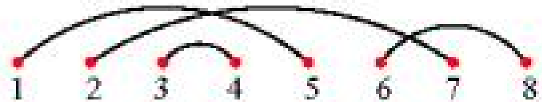
Associate with a matching $M$ on the vertices $1, 2, \ldots, 2n$ a fixed-point free involution $w_M \in \mathfrak{S}_{2n}$:

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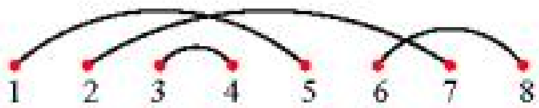
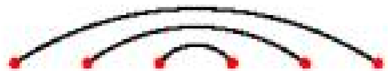


$$w_M = (1,5)(2,7)(3,4)(6,8)$$

**Flaw:** no *symmetry* between is and ds (different distributions on fixed-point free involutions).

3–crossing



3–nesting

$$\boldsymbol{M} = \text{matching}$$
$$\mathbf{cr}(M) = \max\{k : \exists\, k\text{-crossing}\}$$
$$\mathbf{ne}(M) = \max\{k : \exists\, k\text{-nesting}\} = \frac{1}{2}\mathrm{ds}(w_M)$$

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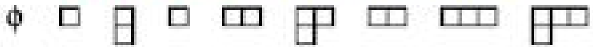
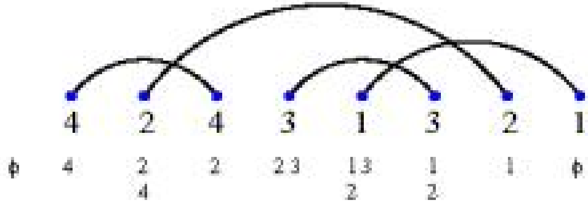
**Theorem.** *Let $f_n(i,j) = \#$ matchings $M$ on $[2n]$ with $\text{cr}(M) = i$ and $\text{ne}(M) = j$. Then* $\boldsymbol{f_n(i,j) = f_n(j,i)}$.

**Corollary.** *$\#$ matchings $M$ on $[2n]$ with $\text{cr}(M) = k$ equals $\#$ matchings $M$ on $[2n]$ with $\text{ne}(M) = k$.*

## Main tool: oscillating tableaux.




shape $(3,1)$, length 8



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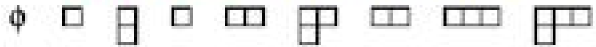


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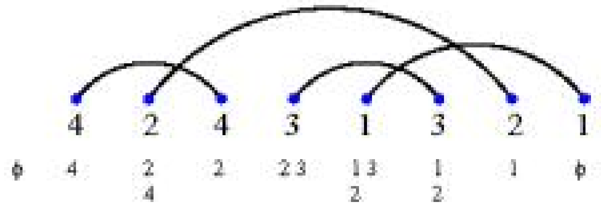


$\Phi$ is a bijection from matchings on $1, 2, \ldots, 2n$ to oscillating tableaux of length $2n$, shape $\emptyset$.

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**Corollary.** *Number of oscillating tableaux of length $2n$, shape $\emptyset$, is $(2n-1)!!$ (related to **Brauer algebra** of dimension $(2n-1)!!$).*

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**Corollary.** *Number of oscillating tableaux of length $2n$, shape $\emptyset$, is $(2n-1)!!$ (related to **Brauer algebra** of dimension $(2n-1)!!$).*

**Schensted's theorem for matchings.** *Let*

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

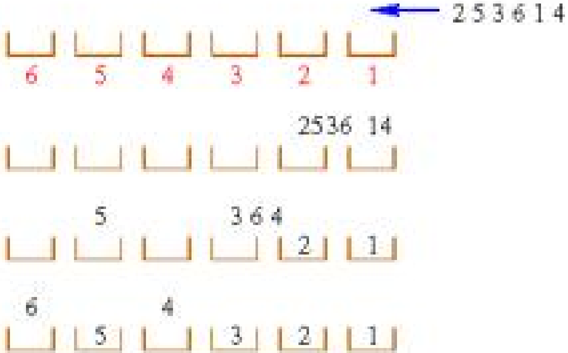
*Then*

$$\mathrm{cr}(M) = \max\{(\lambda^i)'_1 : 0 \leq i \leq n\}$$
$$\mathrm{ne}(M) = \max\{\lambda^i_1 : 0 \leq i \leq n\}.$$

**Proof.** Reduce to ordinary RSK.

**Application:** airplane boarding

**Naive model:** passengers board in order $w = a_1 a_2 \cdots a_n$ for seats $1, 2, \ldots, n$. Each passenger takes one time unit to be seated after arriving at his seat.




**Easy:** Total waiting time $= \mathrm{is}(w)$.

Now let $\mathrm{cr}(M) = i$, $\mathrm{ne}(M) = j$, and

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

Define $M'$ by

$$\Phi(M') = (\emptyset = (\lambda^0)', (\lambda^1)', \ldots, (\lambda^{2n})' = \emptyset).$$

By Schensted's theorem for matchings,

$$\mathrm{cr}(M') = j, \quad \mathrm{ne}(M') = i.$$

Thus $M \mapsto M'$ is an involution on matchings of $[2n]$ interchanging cr and ne.

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$\Rightarrow$ **Theorem.** *Let $f_n(i, j) = \#$ matchings $M$ on $[2n]$ with $\mathrm{cr}(M) = i$ and $\mathrm{ne}(M) = j$. Then* $\boldsymbol{f_n(i, j) = f_n(j, i)}$.

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⇒ **Theorem.** *Let $f_n(i,j) = \#$ matchings $M$ on $[2n]$ with $\mathrm{cr}(M) = i$ and $\mathrm{ne}(M) = j$. Then* $\boldsymbol{f_n(i,j) = f_n(j,i)}$.

**Open:** simple description of $M \mapsto M'$, the analogue of

$$a_1 a_2 \cdots a_n \mapsto a_n \cdots a_2 a_1,$$

which interchanges is and ds.

## Enumeration of $k$-noncrossing matchings (or nestings).

**Recall**: The number of matchings $M$ on $[2n]$ with no crossings, i.e., $\mathrm{cr}(M) = 1$, (or with no nestings) is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

What about the number with $\mathrm{cr}(M) \leq k$?

Assume $\mathrm{cr}(M) \leq k$. Let

$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset)$.

Regard each $\lambda^i = (\lambda^i_1, \ldots, \lambda^i_k) \in \mathbb{N}^k$.

**Corollary.** *The number* $f_k(n)$ *of matchings* $M$ *on* $[2n]$ *with* $\operatorname{cr}(M) \leq k$ *is the number of lattice paths of length* $2n$ *from* $0$ *to* $0$ *in the region*

$$\mathcal{C}_n := \{(a_1, \ldots, a_k) \in \mathbb{N}^k : a_1 \leq \cdots \leq a_k\}$$

*with steps* $\pm e_i$ *(*$e_i$ *=* $i$*th unit coordinate vector).*

$\mathcal{C}_n \otimes \mathbb{R}_{\geq 0}$ is a fundamental chamber for the Weyl group of type $B_k$.

**Grabiner-Magyar**: applied **Gessel-Zeilberger reflection principle** to solve this lattice path problem (not knowing connection with matchings).

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**Theorem.** *Define*

$$\boldsymbol{H_k(x)} = \sum_n f_k(n) \frac{x^{2n}}{(2n)!}.$$

*Then*

$$H_k(x) = \det \left[ I_{|i-j|}(2x) - I_{i+j}(2x) \right]_{i,j=1}^{k}$$

*where*

$$I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}$$

*as before.*

**Example.** $k = 1$ (noncrossing matchings):

$$\begin{aligned} H_1(x) &= I_0(2x) - I_2(2x) \\ &= \sum_{j \geq 0} C_j \frac{x^{2j}}{(2j)!}. \end{aligned}$$

**Example.** $k = 1$ (noncrossing matchings):

$$H_1(x) = I_0(2x) - I_2(2x) = \sum_{j \geq 0} C_j \frac{x^{2j}}{(2j)!}.$$

**Compare:**

$$u_k(n) := \#\{w \in \mathfrak{S}_n : \text{longest increasing subsequence of length } \leq k\}.$$

$$\sum_{n \geq 0} u_k(n) \frac{x^{2n}}{n!^2} = \det \left[ I_{i-j}(2x) \right]_{i,j=1}^k.$$

**Baik-Rains** (implicitly):

$$\lim_{n\to\infty} \text{Prob}\left(\frac{\text{cr}_n(M) - \sqrt{2n}}{(2n)^{1/6}} \le \frac{t}{2}\right) = F_1(t),$$

where

$$\boldsymbol{F_1(t)} = \sqrt{F(t)}\exp\left(\frac{1}{2}\int_t^\infty u(x)dx\right),$$

where $F(t)$ is the Tracy-Widom distribution and $u(x)$ the Painlevé II function.

---

$$F(t) = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x)$$

**Bachmat, et al.:** more sophisticated model

**Two conclusions:**

- Usual system (back-to-front) not much better than random.
- Better: first board window seats, then center, then aisle.

$$g_{j,k}(n) := \#\{\text{matchings } M \text{ on } [2n], \quad \text{cr}(M) \le j,\ \text{ne}(M) \le k\}$$

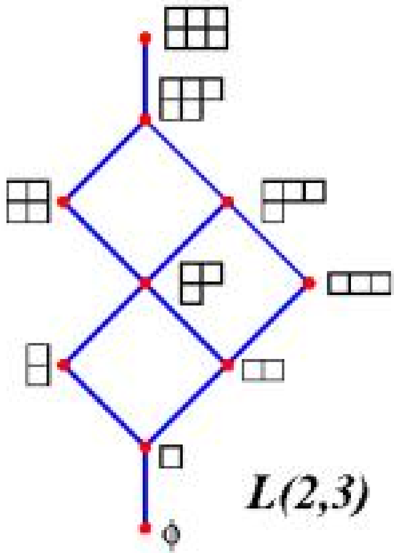
Now

$$g_{j,k}(n) = \#\{(\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset) : \lambda^{i+1} = \lambda^i \pm \square,\ \lambda^i \subseteq j \times k \text{ rectangle}\},$$

a walk on the Hasse diagram $\mathcal{H}(j,k)$ of

$$L(j,k) := \{\lambda \subseteq j \times k \text{ rectangle}\},$$

ordered by inclusion.


L(2,3)

$$A = \text{adjacency matrix of } \mathcal{H}(j,k)$$

$$A_0 = \text{adjacency matrix of } \mathcal{H}(j,k) - \{\emptyset\}.$$

Transfer-matrix method $\Rightarrow$

$$\sum_{n\geq 0} g_{j,k}(n) x^{2n} = \frac{\det(I - xA_0)}{\det(I - xA)}.$$

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**Theorem** (Grabiner, implicitly) Every zero of $\det(I - xA)$ has the form

$$2(\cos(\pi r_1/m) + \cdots + \cos(\pi r_j/m)),$$

where each $r_i \in \mathbb{Z}$ and $m = j + k + 1$.

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where each $r_i \in \mathbb{Z}$ and $m = j + k + 1$.

**Corollary.** *Every factor of* $\det(I - xA)$ *over* $\mathbb{Q}$ *has degree dividing*

$$\frac{1}{2}\phi(2(j + k + 1)),$$

*where* $\phi$ *is the Euler phi-function.*

**Example.**

$j = 2$, $k = 5$, $\frac{1}{2}\phi(16) = 4$:

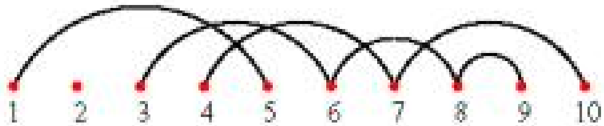
$$\det(I - xA) = (1-2x^2)(1-4x^2+2x^4)$$
$$(1-8x^2+8x^4)(1-8x^2+8x^3-2x^4)$$
$$(1-8x^2-8x^3-2x^4)$$

$j = k = 3$, $\frac{1}{2}\phi(14) = 3$:

$$\det(I-xA) = (1-x)(1+x)(1+x-9x^2-x^3)$$
$$(1-x-9x^2+x^3)(1-x-2x^2+x^3)^2$$
$$(1+x-2x^2-x^3)^2$$

**Partition** of the set $[n]$:

$\{1,5\},\ \{2\},\ \{3,6,8,9\},\ \{4,7,10\}$




Generalize oscillating tableaux to **vacillating tableaux** (related to the **partition algebra**).

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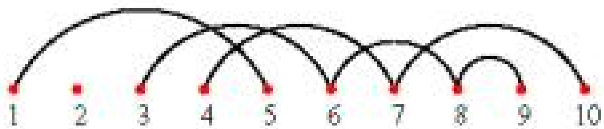


Generalize oscillating tableaux to **vacillating tableaux** (related to the **partition algebra**).

Many other variations: see paper!

**partition** $\lambda \vdash n$: $\lambda = (\lambda_1, \lambda_2, \ldots)$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

$$\sum \lambda_i = n$$