Abstracts of Plenary and Invited Lectures

Section:

0. Plenary Lectures

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Constant Term Identities, Orthogonal Polynomials and Affine Hecke Algebras

Constant term identities

The archetype of the constant term identities to be considered is the following statement, first conjectured in 1961 by F. J. Dyson but proved soon afterwards. Let $x_1, \ldots, x_n$ be independent indeterminates and $k$ a positive integer. Then the constant term in the Laurent polynomial

$$
\Delta_k(x) = \prod_{i \neq j}^{n} \left( \frac{1 - x_i}{x_j} \right)^{k_{i,j}}
$$

is equal to $(nk)!/(k!)^n$.

Later, in 1975, G. E. Andrews proposed an extrapolation of (1) involving an extra parameter $q$. We shall use the notion

$$(x; q)_k = \prod_{i=0}^{k-1} (1 - q^i x).$$

Clearly, as $q \to 1$ we have $(x; q)_k \to (1x)^k$, and $(q; q)_k/(1 - q)^k \to k!$

Let

$$
\Delta_k(x; q) = \prod_{i \neq j}^{n} \left( \varepsilon_{i,j} \frac{x_i}{x_j}; q \right)_k
$$

where $\varepsilon_{i,j} = 1$ or $q$ according as $i < j$ or $i > j$. Then Andrews conjectured that the constant term of (2) (i.e., involving $q$ but none of the $x_i$) should be

$$(q; q)_{nk}/(q; q)_n^n.$$

Clearly (2) reduces to (1) as $q \to 1$, so that Andrews’ conjecture includes Dyson’s.
From our point of view, the polynomial $\Delta_k(x; q)$ is to be regarded as attached to the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ consisting of the $n \times n$ complex matrices with trace zero. In fact each finite-dimensional complex simple Lie algebra $\mathfrak{g}$ gives rise to an analogous Laurent polynomial and constant term identity.

**Orthogonal polynomials**

Let $\Lambda_n$ denote the ring of symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in the field $F = \mathbb{Q}(q)$. For each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leq n$, let $m_\lambda$ denote the monomial symmetric function indexed by $\lambda$, that is to say the sum of all distinct monomials obtainable from $x_1^{\lambda_1}, \ldots, x_n^{\lambda_n}$ by permuting the $x_i$. Clearly the $m_\lambda$ form an $F$-basis of $\Lambda_n$.

We shall use the polynomial $\Delta_k = \Delta_k(x; q)$ to define a scalar product on $\Lambda_n$, as follows: if $f, g \in \Lambda_n$ then

$$\langle f, g \rangle_k = \text{constant term in } f \bar{g} \Delta_k$$

where $\bar{g}$ is obtained from $g$ by replacing each $x_i$ by $x_i^{-1}$. One shows then that there is a unique basis $(P_\lambda)$ of $\Lambda_n$, indexed by partitions of length $\leq n$, such that

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda \mu} m_\mu$$

with coefficients $a_{\lambda \mu} \in F$, where $\mu < \lambda$ means that $\mu$ precedes $\lambda$ in the lexicographic ordering; and

$$\langle P_\lambda, P_\mu \rangle = 0 \quad \text{whenever } \lambda \neq \mu.$$

Moreover, there is a closed formula for the squared norm of $P_\lambda$, namely

$$\langle P_\lambda, P_\lambda \rangle_k = \frac{(t; t)_n}{(1t)^n} \prod_{i,j=1}^n \frac{(q^{\lambda_i - \lambda_j} t^{j-i}; q)_k}{(q^{\lambda_i - \lambda_j} t^{j-i}; q^{-1})_k}$$

where $t = q^k$. In particular, when $\lambda = 0$ we have $P_\lambda = 1$, and (4) then reduces to the constant term (3) of $\Delta_k$.

Again, all this can be done in the context of an arbitrary finite-dimensional simple Lie algebra $\mathfrak{g}$ in place of $\mathfrak{sl}_n$, and thus for each such $\mathfrak{g}$ we have a family of orthogonal polynomials $P_\lambda$, symmetric under the Weyl group of $\mathfrak{g}$ and indexed by the dominant weights.

**Affine Hecke algebras**

In the case of $\mathfrak{sl}_n$ the norm formula (4) can be proved directly. For arbitrary $\mathfrak{g}$ there is an analogous formula which was first proved in full generality by Cherednik. Cherednik’s proof uses the affine Hecke algebra $\mathcal{H}$ attached to $\mathfrak{g}$ and its action on the space spanned by the polynomials $P_\lambda$; this leads to a family of commuting operators on this space whose simultaneous eigenfunctions are precisely the $P_\lambda$. The norm formula (4) – or rather its counterpart for arbitrary $\mathfrak{g}$ – can then be established by induction on $k$. 

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