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## Abstracts of Plenary and Invited Lectures

### Section:

## 0. Plenary Lectures

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**Constant Term Identities, Orthogonal Polynomials and Affine Hecke Algebras**

**Constant term indentities**

The archetype of the constant term identities to be considered is the following statement, first conjectured in 1961 by F. J. Dyson but proved soon afterwards. Let $x_1, \ldots, x_n$ be independent indeterminates and $k$ a positive integer. Then the constant term in the Laurent polynomial

$$\Delta_k(x) = \prod_{\substack{i \neq j \\ i,j=1}}^{n} \left( 1 \frac{x_i}{x_j} \right)^k \tag{1}$$

is equal to $(nk)!/(k!)^n$.

Later, in 1975, G. E. Andrews proposed an extrapolation of (1) involving an extra parameter $q$. We shall use the notion

$$(x;q)_k = \prod_{i=0}^{k-1} \left(1 - q^i x\right).$$

Clearly, as $q \to 1$ we have $(x;q)_k \to (1x)^k$, and $(q;q)_k/(1-q)^k \to k!$

Let

$$\Delta_k(x;q) = \prod_{\substack{i \neq j \\ i,j=1}}^{n} \left( \varepsilon_{ij} \frac{x_i}{x_j}; q \right)_k \tag{2}$$

where $\varepsilon_{ij} = 1$ or $q$ according as $i < j$ or $i > j$. Then Andrews conjectured that the constant term of (2) (i. e. , involving $q$ but none of the $x_i$) should be

$$(q;q)_{nk}/(q;q)_k^n. \tag{3}$$

Clearly (2) reduces to (1) as $q \to 1$, so that Andrews' conjecture includes Dyson's.

From our point of view, the polynomial $\Delta_k(x;q)$ is to be regarded as attached to the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ consisting of the $n \times n$ complex matrices with trace zero. In fact each finite-dimensional complex simple Lie algebra $\mathfrak{g}$ gives rise to an analogous Laurent polynomial and constant term identity.

**Orthogonal polynomials**

Let $\Lambda_n$ denote the ring of symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in the field $F = \mathbb{Q}(q)$. For each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leq n$, let $m_\lambda$ denote the monomial symmetric function indexed by $\lambda$, that is to say the sum of all distinct monomials obtainable from $x_1^{\lambda_1}, \ldots, x_n^{\lambda_n}$ by permuting the $x_i$, Clearly the $m_\lambda$ form an $F$-basis of $\Lambda_n$.

We shall use the polynomial $\Delta_k = \Delta_k(x;q)$ to define a scalar product on $\Lambda_n$, as follows: if $f, g \in \Lambda_n$ then

$$\langle f, g \rangle_k = \text{ constant term in } f\bar{g}\Delta_k$$

where $\bar{g}$ is obtained from $g$ by replacing each $x_i$ by $x_i^{-1}$. One shows then that there is a unique basis $(P_\lambda)$ of $\Lambda_n$, indexed by partitions of length $\leq n$, such that

$$\text{(a)} \qquad P_\lambda = m_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} m_\mu$$

with coefficients $a_{\lambda\mu} \in F$, where $\mu < \lambda$ means that $\mu$ precedes $\lambda$ in the lexicographic ordering; and

$$\text{(b)} \qquad \langle P_\lambda, P_\mu \rangle = 0 \quad \text{whenever } \lambda \neq \mu.$$

Moreover, there is a closed formula for the squared norm of $P_\lambda$, namely

$$\langle P_\lambda, P_\lambda \rangle_k = \frac{(t;t)_n}{(1t)^n} \prod_{\substack{i<j \\ i,j=1}}^{n} \frac{(q^{\lambda_i - \lambda_j} t^{j-i}; q)_k}{(q^{\lambda_i \lambda_j} t^{j-i}; q^{-1})_k} \tag{4}$$

where $t = q^k$. In particular, when $\lambda = 0$ we have $P_\lambda = 1$, and (4) then reduces to the constant term (3) of $\Delta_k$.

Again, all this can be done in the context of an arbitrary finite-dimensional simple Lie algebra $\mathfrak{g}$ in place of $\mathfrak{sl}_n$, and thus for each such $\mathfrak{g}$ we have a family of orthogonal polynomials $P_\lambda$, symmetric under the Weyl group of $\mathfrak{g}$ and indexed by the dominant weights.

**Affine Hecke algebras**

In the case of $\mathfrak{sl}_n$ the norm formula (4) can be proved directly. For arbitrary $\mathfrak{g}$ there is an analogous formula which was first proved in full generality by Cherednik. Cherednik's proof uses the affine Hecke algebra $\mathcal{H}$ attached to $\mathfrak{g}$ and its action on the space spanned by the polynomials $P_\lambda$; this leads to a family of commuting operators on this space whose simultaneous eigenfunctions are precisely the $P_\lambda$. The norm formula (4) – or rather its counterpart for arbitrary $\mathfrak{g}$ – can then be established by induction on $k$.